A posteriori error estimates, stopping criteria, and adaptivity for multiphase compositional Darcy flows in porous media

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Outline

1. Model and discretization

2. A posteriori error estimates and adaptive resolution

3. Numerical results
In the context of petroleum reservoir engineering, numerical simulation is used to predict oil production and to plan exploitation. Modelling the flow of several fluids through a porous medium leads to highly nonlinear and computationally expensive problems. 

**Goal** significantly reduce simulation time

**Idea** use a posteriori error estimators to make smart online choices
In a nutshell II

Fix $M^0$ and $\tau_0$. Set $t^0 \leftarrow 0$, $n \leftarrow 0$ and set the initial solution $X^0_{h\tau}$.

while $t^n \leq t^F$ do

Set $n \leftarrow n + 1$, $M^n \leftarrow M^{n-1}$, $\tau^n \leftarrow \tau^{n-1}$.

repeat

Equilibration of spatial and temporal errors

Set $k \leftarrow 0$ and $X^{n,0}_{h\tau} \leftarrow X^{n-1}_{h\tau}$.

repeat

Newton iterations

$k \leftarrow k + 1$ and $i \leftarrow 0$. Set $X^{n,k,0}_{h\tau} \leftarrow X^{n,k-1}_{h\tau}$.

Set up the linear system

repeat

Algebraic iterations

Set $i = i + 1$ and perform one iteration of the algebraic solver

Compute $\eta_{n,k,i}^{\text{sp}}$, $\eta_{n,k,i}^{\text{tm}}$, $\eta_{n,k,i}^{\text{lin}}$, $\eta_{n,k,i}^{\text{alg}}$

until stopping criterion

until stopping criterion

Adapt the time step $\tau^n$

until time-space equilibration

Set $X^{n}_{h\tau} \leftarrow X^{n,k,i}_{h\tau}$, and $t^n \leftarrow t^{n-1} + \tau^n$.

end while
In a nutshell III

Figure: Cumulated linear solver iteration as a function of stopping criteria
Let two sets of phases $\mathcal{P}$ and components $\mathcal{C}$ be given. We define for all $p \in \mathcal{P}$ and all $c \in \mathcal{C}$ the relevant subsets:

$$\mathcal{C}_p := \{c \in \mathcal{C}; \ c \text{ is present in } p\}, \quad \mathcal{P}_c := \{p \in \mathcal{P}; \ c \text{ is present in } p\}$$

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**Figure**: Example of a two-phase, three-component flux.
The compositional Darcy model II

- Formulation inspired by [Coats, 1980, Eymard et al., 2012]
- The unknowns of the model are

\[
\mathcal{X} := \begin{pmatrix}
    P \\
    (S_p)_{p \in \mathcal{P}} \\
    (C_{p,c})_{p \in \mathcal{P}, c \in \mathcal{C}_p}
\end{pmatrix} = \begin{pmatrix}
    P \\
    S \\
    (C_p)_{p \in \mathcal{P}}
\end{pmatrix}
\]

- We define for all \( p \in \mathcal{P} \) the phase pressure as

\[
P_p(P, S) = P + P_{c_p}(S)
\]

- The (average) phase velocity is given by Darcy’s law

\[
\mathbf{v}_p = -\Lambda \left( \nabla P_p + \rho_p g \nabla z \right)
\]
Let \( \Omega \) denote the space domain and \( t_F > 0 \) the simulation time.

**Conservation of the quantity of matter:** For all \( c \in \mathcal{C} \),

\[
\partial_t l_c + \nabla \cdot \Phi_c = q_c \quad \text{in } \Omega \times (0, t_F)
\]

with \( q_c \) source piecewise constant in space-time and

\[
l_c := \phi \sum_{p \in \mathcal{P}_c} \zeta_p S_p C_{p,c}, \quad \Phi_c := \sum_{p \in \mathcal{P}_c} \{ \Phi_{p,c} := \nu_p C_{p,c} \mathbf{v}_p \},
\]

**Initial and boundary conditions**

\[
l_c(0) = l^0_c \quad \text{in } \Omega, \quad \Phi_c \cdot \mathbf{n}_\Omega = 0 \quad \text{on } \partial \Omega \times (0, t_F)
\]
The compositional Darcy model IV

- Saturation of the pore volume

\[
\sum_{p \in P} S_p = 1
\]

- Partition of the matter into components

\[
\sum_{c \in C_p} C_{p,c} = 1 \quad \forall p \in P
\]

- Thermodynamic equilibrium relations close the system
A classical finite volume scheme I

- We consider a popular fully implicit finite volume discretization.
- The numerical fluxes are based on phase-upwind and two-point fluxes.
- On phase-upwind cf., e.g., [Brenier and Jaffré, 1991].
A classical finite volume scheme II

- We consider a partition \((t^n)_{0 \leq n \leq N}\) of \((0, t_F)\) with

\[ t^n = \sum_{i=1}^{n} \tau_i \quad \tau_i > 0 \quad \forall 1 \leq i \leq n \]

- We denote by \((\mathcal{M}^n)_{0 \leq n \leq N}\) a sequence of meshes of \(\Omega\) with

\[ \mathcal{M}^n = \{M\} \]

- The discrete unknowns are, for all \(M \in \mathcal{M}^n\),

\[ \mathcal{X}^n_{\mathcal{M}} := (\mathcal{X}^n_M)_{M \in \mathcal{M}^n}, \quad \mathcal{X}^n_M := \begin{pmatrix} P^n_M \\ S^n_M \\ (C^n_{p,M})_{p \in \mathcal{P}} \end{pmatrix} \]
For each phase, the **discrete phase pressure** is given by

\[
P^n_{p,M}(P^n_M, S^n_M) = P^n_M + P_{cp}(S^n_M) \quad \forall M \in \mathcal{M}^n
\]

The **discrete phase velocity** is given by, for all \(\sigma \subset \partial M \cap \partial L\),

\[
F^n_{p,M,\sigma}(\chi^n_M) := |\sigma| \frac{\alpha_M \alpha_L}{\alpha_M + \alpha_L} \left[ P^n_{p,M} - P^n_{p,L} + \rho^n_{p,\sigma} g(z_M - z_L) \right],
\]

while \(F^n_{p,M,\sigma}(\chi^n_M) = 0\) if \(\sigma \subset \partial \Omega\)
Discrete conservation of the quantity of matter: For all $c \in C$,

$$|M| \partial_t l_{c,M} + \sum_{\sigma \in E_M} F_{c,M,\sigma}(\mathcal{X}_M^n) = |M| q_{c,M}^n \quad \forall M \in \mathcal{M}^n$$

with

$$l_{c,M}^n = \phi \sum_{p \in \mathcal{P}_c} \zeta_p(P_{p,M}, C_{p,M}^n) S_{p,M}^n C_{p,c,M}^n$$

and molar component flux

$$F_{c,M,\sigma}(\mathcal{X}_M^n) := \sum_{p \in \mathcal{P}_c} \left\{ F_{p,c,M,\sigma}(\mathcal{X}_M^n) := \nu_p C_{p,c,M}^n \right\}$$

Closure laws are enforced cell-wise
Outline

1. Model and discretization

2. A posteriori error estimates and adaptive resolution

3. Numerical results
Essential bibliography

- A posteriori estimates for model unsteady nonlinear problems
- A posteriori estimates for degenerate parabolic problems
  - [Nochetto et al., 2000, Ohlberger, 2001]
  - [DP et al., 2012, DP et al., 2013]
- Adaptive mesh refinement in reservoir simulation
  - [Heinemann, 1983, Ewing et al., 1989] and many more
  - [Mamaghani et al., 2011] (SAGD with $C = 2$, $P = 3$)
- Smart online choices
  - [Jiránek et al., 2010] (stopping criteria)
  - [Ern and Vohralík, 2013] (inexact Newton)
  - [DP et al., 2012] (stopping criteria + parameter selection)
Assumption (Weak solution)

There exists a weak solution $X$ such that

- For all $p \in P$, $P_p(P, S) \in X := L^2(H^1(\Omega))$
- For all $c \in C$, $l_c \in Y := H^1(L^2(\Omega))$ and $\Phi_c \in [L^2(L^2(\Omega))]^d$
- The following equality holds for all $\varphi \in X$ and all $c \in C$:

$$
\int_0^{t_F} \left\{ (\partial_t l_c, \varphi)(t) - (\Phi_c, \nabla \varphi)(t) \right\} dt = \int_0^{t_F} (q_c, \varphi)(t) dt
$$

- The initial condition and the closure equations hold

We equip the space $X$ with the norm

$$
\|\varphi\|_X := \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{M \in \mathcal{M}^n} \|\varphi\|_{X,M}^2 dt \right\}^{1/2}, \quad \|\varphi\|_{X,M}^2 := \varepsilon h_M^{-2} \|\varphi\|_{M}^2 + \|\nabla \varphi\|_{M}^2
$$
Fully computable upper bound II

\[ \mathcal{N} := \left\{ \sum_{c \in \mathcal{C}} \mathcal{N}_c^2 \right\}^{1/2} + \left\{ \sum_{p \in \mathcal{P}} \mathcal{N}_p^2 \right\}^{1/2} \]

- Dual norm of the residual

\[ \mathcal{N}_c := \sup_{\varphi \in X, \|\varphi\|_X = 1} \int_0^{t_F} \left\{ (\partial_t l_c - \partial_t l_{c,h\tau}, \varphi)(t) - (\Phi_c - \Phi_{c,h\tau}, \nabla \varphi)(t) \right\} dt \]

with \( \Phi_{c,h\tau} = \sum_{p \in \mathcal{P}_c} \nu_p(P_{p,h\tau}, S_{h\tau}, C_{p,h\tau})C_{p,c,h\tau} v_p(P_{p,h\tau}, C_{p,h\tau}) \)

- Nonconformity in \( X \):

\[ \mathcal{N}_p := \inf_{\varphi_p \in X} \left\{ \sum_{c \in \mathcal{C}_p} \int_0^{t_F} \| \Psi_{p,c}(P_{p,h\tau})(t) - \Psi_{p,c}(\varphi_p)(t) \|^2 dt \right\}^{1/2} \]

with \( \Psi_{p,c}(\varphi) := \nu_p(P_{p,h\tau}, S_{h\tau}, C_{p,h\tau})C_{p,c,h\tau} \Lambda \nabla \varphi \)
Theorem (Fully computable upper bound)

The following guaranteed upper bounds hold:

\[ \mathcal{N}_c \leq \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{M \in \mathcal{M}^n} \left( \eta_{R,M,c}^n + \eta_{F,M,c}^n(t) \right)^2 \, dt \right\}^{1/2} \quad \forall c \in \mathcal{C}, \]

\[ \mathcal{N}_p \leq \left\{ \sum_{c \in \mathcal{C}_p} \sum_{n=1}^{N} \int_{I_n} \sum_{M \in \mathcal{M}^n} \left( \eta_{NC,M,p,c}^n(t) \right)^2 \, dt \right\}^{1/2} \quad \forall p \in \mathcal{P}, \]

with estimators given by, for all \( c \in \mathcal{C} \) and all \( M \in \mathcal{M}^n \),

\[ \eta_{R,M,c}^n := \tilde{C}_{P,M} h_M \| q_{c,h}^n - \partial_t l_{c,h\tau} - \nabla \cdot \Theta_{c,h}^n \|_M, \]

\[ \eta_{F,M,c}^n(t) := \| \Theta_{c,h}^n - \Phi_{c,h\tau}(t) \|_M, \]

\[ \eta_{NC,M,p,c}^n(t) := \| \Psi_{p,c}(P_{p,h\tau})(t) - \Psi_{p,c}(\mathcal{P}_{p,h\tau})(t) \|_M \quad \forall p \in \mathcal{P}_c. \]

where, for all \( c \in \mathcal{C}, \Theta_{c,h}^n \in \text{RTN}(\mathcal{M}^n) \) s.t.

\[ (q_{c,h}^n - \partial_t l_{c,h\tau} - \nabla \cdot \Theta_{c,h}^n, 1)_M = 0 \quad \forall M \in \mathcal{M}^n. \]
Solving the discrete problem amounts to zeroing the residuals

\[ R^n_{c,M}(\mathcal{X}_M^n) := |M| \frac{l_c,M(\mathcal{X}_M^n) - l^n_{c,M}}{\tau^n} + \sum_{\sigma \in \mathcal{E}^{i,n}_M} F_{c,M,\sigma}(\mathcal{X}_M^n) - |M|q^n_{c,M} = 0 \]

The Newton method generates a sequence \((\mathcal{X}_M^{n,k})_{k \geq 0}\) by solving

\[ \sum_{M' \in \mathcal{M}^n} \frac{\partial R^n_{c,M}}{\partial \mathcal{X}_M^{n,k-1}} \cdot (\mathcal{X}_M^{n,k} - \mathcal{X}_M^{n,k-1}) + R^n_{c,M}(\mathcal{X}_M^{n,k-1}) = 0 \]
The resulting linear system can be solved by an iterative linear solver.

The residuals at Newton iteration \( k \) and linear solver iteration \( i \) read

\[
R_{c,M}^{n,k,i} = |M| \left( l_{c,M} (x_{M}^{n,k-1}) + \mathcal{L}_{c,M}^{n,k,i} - l_{c,M}^{n-1} \right) \left( \frac{1}{\tau^n} \right) + \sum_{\sigma \in \mathcal{E}_{M}^{i,n}} F_{c,M,\sigma}^{n,k,i} - |M| q_{c,M}^{n},
\]

where \( F_{c,M,\sigma}^{n,k,i} \) is a linearized component flux.
Corollary (Time-localized a posteriori error estimate)

For a given time step $n$, Newton iteration $k$, and linear iteration $i$ we have

$$
\mathcal{N}^n_{c} \leq \left\{ \int_{I_n} \sum_{M \in \mathcal{M}^n} (\eta^{n,k,i}_{R,M,c} + \eta^{n,k,i}_{F,M,c}(t) + \eta^{n,k,i}_{NA,M,c})^2 dt \right\}^{1/2} \quad \forall c \in \mathcal{C},
$$

$$
\mathcal{N}^n_{p} \leq \left\{ \sum_{c \in \mathcal{C}_p} \int_{I_n} \sum_{M \in \mathcal{M}^n} (\eta^{n,k,i}_{NC,M,p,c}(t))^2 dt \right\}^{1/2} \quad \forall p \in \mathcal{P},
$$

where $\eta^{n,k,i}_{NA,M,c}$ is related to the nonlinear accumulation term.
We decompose the component flux reconstruction $\Theta_{c,h}^{n,k,i} \in \text{RTN}(\mathcal{M}^n)$ as

$$
\Theta_{c,h}^{n,k,i} := \Theta_{\text{disc},c,h}^{n,k,i} + \Theta_{\text{lin},c,h}^{n,k,i} + \Theta_{\text{alg},c,h}^{n,k,i}
$$

The discretization flux reconstruction $\Theta_{\text{disc},c,h}^{n,k,i} \in \text{RTN}(\mathcal{M}^n)$ is s.t.

$$
(\Theta_{\text{disc},c,h}^{n,k,i} \cdot n_M, 1)_\sigma := F_{c,M,\sigma}(X_{\mathcal{M}}^{n,k,i})
$$

The linearization error flux reconstruction $\Theta_{\text{lin},c,h}^{n,k,i} \in \text{RTN}(\mathcal{M}^n)$ is s.t.

$$
(\Theta_{\text{lin},c,h}^{n,k,i} \cdot n_M, 1)_\sigma = F_{c,M,\sigma}^{n,k,i} - F_{c,M,\sigma}(X_{\mathcal{M}}^{n,k,i})
$$

The algebraic error flux reconstruction $\Theta_{\text{alg},c,h}^{n,k,i} \in \text{RTN}(\mathcal{M}^n)$ is s.t.

$$
(\Theta_{\text{alg},c,h}^{n,k,i} \cdot n_M, 1)_{\partial M} := -R_{c,M}^{n,k,i}
$$
Distinguishing the error components II

- **Space error estimator**

\[
\eta_{sp,M,c}^{n,k,i}(t) := \eta_{R,M,c}^{n,k,i} + \| \Theta_{\text{disc},c,h}^{n,k,i} - \Phi_{c,h\tau}^{n,k,i}(t^n) \| M + \left\{ \sum_{p \in \mathcal{P}_c} (\eta_{NC,M,p,c}^{n,k,i}(t))^2 \right\}^{1/2}
\]

- **Time error estimator**

\[
\eta_{tm,M,c}^{n,k,i}(t) := \| \Phi_{c,h\tau}^{n,k,i}(t^n) - \Phi_{c,h\tau}^{n,k,i}(t) \| M
\]

- **Linearization error estimator**

\[
\eta_{\text{lin},M,c}^{n,k,i} := \| \Theta_{\text{lin},c,h}^{n,k,i} M \| + \eta_{\text{NA},M,c}^{n,k,i}
\]

- **Algebraic error estimator**

\[
\eta_{\text{alg},M,c}^{n,k,i} := \| \Theta_{\text{alg},c,h}^{n,k,i} \| M
\]
Distinguishing the error components III

**Corollary (Distinguishing the different error components)**

The following estimate holds:

\[
N^n \leq \left\{ \sum_{c \in C} \left( \eta_{sp,c}^{n,k,i} + \eta_{tm,c}^{n,k,i} + \eta_{lin,c}^{n,k,i} + \eta_{alg,c}^{n,k,i} \right)^2 \right\}^{1/2},
\]

with estimators given by

\[
\eta_{sp,c}^{n,k,i} := \left\{ 4 \int_{I_n} \sum_{M \in \mathcal{M}^n} \left( \eta_{sp,M,c}^{n,k,i}(t) \right)^2 dt \right\}^{1/2},
\]

\[
\eta_{tm,c}^{n,k,i} := \left\{ 2 \int_{I_n} \sum_{M \in \mathcal{M}^n} \left( \eta_{tm,M,c}^{n,k,i}(t) \right)^2 dt \right\}^{1/2},
\]

\[
\eta_{lin,c}^{n,k,i} := \left\{ 2 \tau^n \sum_{M \in \mathcal{M}^n} \left( \eta_{lin,M,c}^{n,k,i} \right)^2 \right\}^{1/2},
\]

\[
\eta_{alg,c}^{n,k,i} := \left\{ 2 \tau^n \sum_{M \in \mathcal{M}^n} \left( \eta_{alg,M,c}^{n,k,i} \right)^2 \right\}^{1/2}.
\]
A fully adaptive algorithm

Fix $M^0$ and $\tau_0$. Set $t^0 \leftarrow 0$, $n \leftarrow 0$ and set the initial solution $X^0_{h\tau}$.

while $t^n \leq t^F$ do

Set $n \leftarrow n + 1$, $M^n \leftarrow M^{n-1}$, $\tau^n \leftarrow \tau^{n-1}$.

repeat \{Equilibration of spatial and temporal errors\}

Set $k \leftarrow 0$ and $X^{n,0}_{h\tau} \leftarrow X_{h\tau}^{n-1}$.

repeat \{Newton iterations\}

$k \leftarrow k + 1$ and $i \leftarrow 0$. Set $X^{n,k,0}_{h\tau} := X_{h\tau}^{n,k-1}$.

Set up the linear system

repeat \{Algebraic iterations\}

Set $i = i + 1$ and perform one iteration of the algebraic solver

Compute $\eta_{sp}^{n,k,i}$, $\eta_{tm}^{n,k,i}$, $\eta_{lin}^{n,k,i}$, $\eta_{alg}^{n,k,i}$

until

until

Adapt the time step $\tau^n$

until

Set $X^{n}_{h\tau} \leftarrow X^{n,k,i}_{h\tau}$, and $t^n \leftarrow t^{n-1} + \tau^n$.

end while
A fully adaptive algorithm

Fix $M^0$ and $\tau_0$. Set $t^0 \leftarrow 0$, $n \leftarrow 0$ and set the initial solution $X^0_{h\tau}$.

while $t^n \leq t^F$ do

Set $n \leftarrow n + 1$, $M^n \leftarrow M^{n-1}$, $\tau^n \leftarrow \tau^{n-1}$.

repeat {Equilibration of spatial and temporal errors}

Set $k \leftarrow 0$ and $X^{n,0}_{h\tau} \leftarrow X^{n-1}_{h\tau}$.

repeat {Newton iterations}

$k \leftarrow k + 1$ and $i \leftarrow 0$. Set $X^{n,k,0}_{h\tau} := X^{n,k-1}_{h\tau}$.

Set up the linear system

repeat {Algebraic iterations}

Set $i = i + 1$ and perform one iteration of the algebraic solver

Compute $\eta_{sp}^{n,k,i}$, $\eta_{tm}^{n,k,i}$, $\eta_{lin}^{n,k,i}$, $\eta_{alg}^{n,k,i}$

until $\eta_{alg,c}^{n,k,i} \leq \gamma_{alg}(\eta_{sp,c}^{n,k,i} + \eta_{tm,c}^{n,k,i} + \eta_{lin,c}^{n,k,i}) \quad \forall c \in C$

until

Adapt the time step $\tau^n$

until

Set $X^n_{h\tau} \leftarrow X^{n,k,i}_{h\tau}$, and $t^n \leftarrow t^{n-1} + \tau^n$.

end while
A fully adaptive algorithm

Fix $\mathcal{M}^0$ and $\tau_0$. Set $t^0 \leftarrow 0$, $n \leftarrow 0$ and set the initial solution $\mathcal{X}_{h\tau}^0$.

while $t^n \leq t^F$ do

Set $n \leftarrow n + 1$, $\mathcal{M}^n \leftarrow \mathcal{M}^{n-1}$, $\tau^n \leftarrow \tau^{n-1}$.

repeat {Equilibration of spatial and temporal errors}

Set $k \leftarrow 0$ and $\mathcal{X}_{h\tau}^{n,0} \leftarrow \mathcal{X}_{h\tau}^{n-1}$.

repeat {Newton iterations}

$k \leftarrow k + 1$ and $i \leftarrow 0$. Set $\mathcal{X}_{h\tau}^{n,k,0} := \mathcal{X}_{h\tau}^{n,k-1}$.

Set up the linear system

repeat {Algebraic iterations}

Set $i = i + 1$ and perform one iteration of the algebraic solver

Compute $\eta_{\text{sp},i}^{n,k,i}$, $\eta_{\text{tm},i}^{n,k,i}$, $\eta_{\text{lin},i}^{n,k,i}$, $\eta_{\text{alg},i}^{n,k,i}$

until $\eta_{\text{alg},i}^{n,k,i} \leq \gamma_{\text{alg}}(\eta_{\text{sp},c}^{n,k,i} + \eta_{\text{tm},c}^{n,k,i} + \eta_{\text{lin},c}^{n,k,i})$ $\forall c \in \mathcal{C}$

until $\eta_{\text{lin},i}^{n,k,i} \leq \gamma_{\text{lin}}(\eta_{\text{sp},c}^{n,k,i} + \eta_{\text{tm},c}^{n,k,i})$ $\forall c \in \mathcal{C}$

Adapt the time step $\tau^n$

until

Set $\mathcal{X}_{h\tau}^{n} \leftarrow \mathcal{X}_{h\tau}^{n,k,i}$, and $t^n \leftarrow t^{n-1} + \tau^n$.

end while
A fully adaptive algorithm

Fix $M^0$ and $\tau_0$. Set $t^0 \leftarrow 0$, $n \leftarrow 0$ and set the initial solution $\chi^0_{h\tau}$.

while $t^n \leq t^F$ do

Set $n \leftarrow n + 1$, $M^n \leftarrow M^{n-1}$, $\tau^n \leftarrow \tau^{n-1}$.

repeat {Equilibration of spatial and temporal errors}

Set $k \leftarrow 0$ and $\chi^{n,0}_{h\tau} \leftarrow \chi^{n-1}_{h\tau}$.

repeat {Newton iterations}

$k \leftarrow k + 1$ and $i \leftarrow 0$. Set $\chi^{n,k,0}_{h\tau} := \chi^{n,k-1}_{h\tau}$.

Set up the linear system

repeat {Algebraic iterations}

Set $i = i + 1$ and perform one iteration of the algebraic solver

Compute $\eta_{sp,k,i}^n$, $\eta_{tm,k,i}^n$, $\eta_{lin,k,i}^n$, $\eta_{alg,k,i}^n$

until $\eta_{alg,k,i}^n \leq \gamma_{alg}(\eta_{sp,k,i}^n + \eta_{tm,k,i}^n + \eta_{lin,k,i}^n) \quad \forall c \in C$

until $\eta_{lin,k,i}^n \leq \gamma_{lin}(\eta_{sp,k,i}^n + \eta_{tm,k,i}^n) \quad \forall c \in C$

Adapt the time step $\tau^n$

until $\gamma_{tm} \eta_{sp,k,i}^n \leq \eta_{tm,k,i}^n \leq \Gamma_{tm} \eta_{sp,k,i}^n \quad \forall c \in C$

Set $\chi^{n,k,i}_{h\tau} \leftarrow \chi^{n,k,i}_{h\tau}$, and $t^n \leftarrow t^{n-1} + \tau^n$.

end while
Adaptive stopping criteria

Figure: Evolution of the error component estimators for a fixed mesh as a function of GMRes (left) and Newton iterations (right)
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A numerical example

Figure: Numerical example: injection of a mixture of CO$_2$ and N$_2$ into a reservoir initially saturated with C$_7$H$_{16}$
Figure: Liquid saturation, classical (left) vs. adaptive (right) resolution at times $7.8 \times 10^7$ s and $2.1 \times 10^8$ s
Figure: Cumulated oil production, classical vs. adaptive resolution
Figure: Newton iterations at each time step (left) and cumulated number of Newton iterations as a function of time (right).
Homogeneous medium IV

**Figure**: GMRes iterations for each Newton iteration (left) and cumulated as a function of time (right)
Figure: Liquid saturation, classical (left) and adaptive (right) resolutions at times $5.2 \times 10^7$ s, $1.04 \times 10^8$ s, and $1.6 \times 10^8$ s (heterogeneous medium)
Figure: Cumulated Newton (left) and GMRes (right) iterations as a function of time (right) (heterogeneous medium)
Upstream differencing for multiphase flow in reservoir simulation.

An equation of state compositional model.

A posteriori error estimates, stopping criteria, and adaptivity for multiphase compositional Darcy flows in porous media.
Submitted. Preprint hal-00839487.

Adaptive regularization, linearization, and discretization and a posteriori error control for the two-phase Stefan problem.
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Adaptive finite element methods for parabolic problems. IV. Nonlinear problems.

Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs.
DOI 10.1137/120896918.

Efficient use of locally refined grids for multiphase reservoir simulation.
*Society of Petroleum Engineers*.

Vertex-centred discretization of multiphase compositional darcy flows on general meshes.
Using local grid refinement in a multiple-application reservoir simulator.
*Society of Petroleum Engineers.*

A posteriori error estimates including algebraic error and stopping criteria for iterative solvers.

Development of a refinement criterion for adaptive mesh refinement in steam-assisted gravity drainage simulation.

A posteriori error estimation and adaptivity for degenerate parabolic problems.

A posteriori error estimate for finite volume approximations to singularly perturbed nonlinear convection–diffusion equations.

A posteriori error estimates for nonlinear problems. $L^r(0, T; L^p(\Omega))$-error estimates for finite element discretizations of parabolic equations.

A posteriori error estimates for nonlinear problems: $L^r(0, T; W^{1,p}(\Omega))$-error estimates for finite element discretizations of parabolic equations.