Méthode d’équation intégrale accélérée par la méthode multipôle rapide pour les problèmes tridimensionnels de propagation d’ondes élastiques dans un demi-espace

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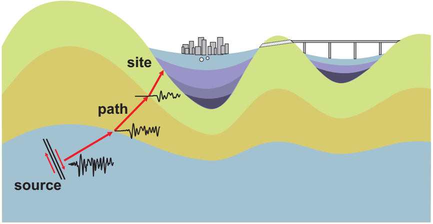
Outline

1. Motivations and background
2. Formulation of the Elastic Half-space Fundamental solutions
3. New Fast Multipole Formulation for the Elastic Half-space
4. Numerical experiments and complexity study
5. Conclusions and future work

Motivation

Modelling of elastic wave propagation in large/unbounded domains



- Site effects
- Soil-structure interaction
- Computational forward solution method for inverse problems

Pros and cons of Boundary Element Methods for seismic waves

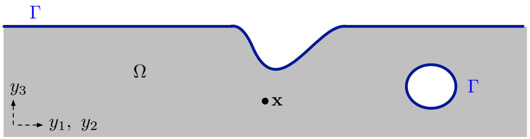
Domain methods (FEM, SEM, ...)
→ Domain mesh
→ Approx. radiation conditions
→ Sparse matrix

BEM
→ Surface mesh (i.e. reduced dim.)
→ Exact radiation conditions
→ Fully-populated matrix

→ BEM adequate for large (unbounded) media, simple (linear) prop
→ Fully-populated BEM influence matrix: severe limiting factor

Formulation with the **full-space** fundamental solutions (F.S.)

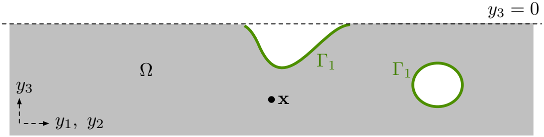
$$c_{ik}(\mathbf{x})u_i(\mathbf{x}) = \int_\Gamma [t_i(\mathbf{y})U_i^k(\mathbf{x},\mathbf{y};\omega) - u_i(\mathbf{y})T_i^k(\mathbf{x},\mathbf{y};\omega)]\,\mathrm{d}S_y, \ (x \in \partial\Omega)$$



BIE for half-space problems

Formulation with the **half-space** fundamental solutions (F.S.)

$$c_{ik}(\mathbf{x})u_i(\mathbf{x}) = \int_{\Gamma_1} [t_i(\mathbf{y})U_i^{HS,k}(\mathbf{x},\mathbf{y};\omega) - u_i(\mathbf{y})T_i^{HS,k}(\mathbf{x},\mathbf{y};\omega)]\, \mathrm{d}S_y, \ (x \in \partial\Omega)$$



U^{HS} and T^{HS} satisfy traction-free condition on free surface $y_3 = 0$

Numerical advantages and issues

+ No truncation issue (Γ_1 is a bounded surface)
+ $\#$ DOFs drastically reduced compared to full-space F.S.

– Closed-form of U^{HS} and T^{HS} not available
– Substantial CPU time (integrals of oscillating functions over infinite intervals)

Fast Multipole accelerated BEM

Fast Multipole Method (FMM)

- Accelerates standard BEM and reduces memory requirements
- Fast, approximate method for evaluating the linear integral operator (matrix-vector product, called by iterative solver)

Previous results in elastodynamics for **full-space** F.S.

- Multi-level fast multipole method successfully extended to single-region 3-D frequency-domain elastodynamics[1] and viscoelastodynamics[2]
- Multi-region FMM implemented for 3-D frequency-domain viscoelastodynamics (based on single-region FMM)[3]

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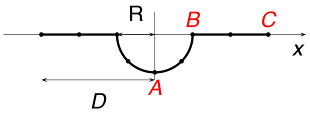
[1] [Chaillat *et al.*, CMAME, 2008]
[2] [Grasso *et al.*, EABE, 2012]
[3] [Chaillat *et al.*, GJI, 2009]

Numerical results in elastodynamics for **full-space** F.S.

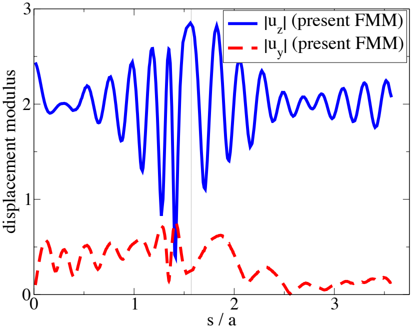
- Implementation (code COFFEE) consistent with theoretical complexity $O(N \log N)$, BE models of size $N = O(10^6)$
- Implementation tested against published results for low frequencies, and run for higher frequencies

Diffraction plane P wave by semi-spherical canyon $k_P a/\pi = 5$

$N = 287\,946$ (86 iter., 162 s per iter., single-proc. 3 GHz PC)



s/a: normalized arc-length coordinate along ABC



Our contribution for **half-space** F.S.

Approach to define a FMM for **full-space** elastodynamics F.S.

- Full-space elastodynamics F.S. are derivatives of Helmholtz F.S.

$$
\begin{aligned}
U_i^k(\mathbf{x},\mathbf{y};\omega) &= \frac{1}{k_S^2\mu}\left((\delta_{qs}\delta_{ik}-\delta_{qk}\delta_{is})\frac{\partial}{\partial x_q}\frac{\partial}{\partial y_s}G_S(|\mathbf{x}-\mathbf{y}|)+\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_k}G_P(|\mathbf{x}-\mathbf{y}|)\right)\\
T_i^k(\mathbf{x},\mathbf{y};\omega) &= C_{ijh\ell}\frac{\partial}{\partial y_\ell}U_h^k(\mathbf{x},\mathbf{y};\omega)n_j(\mathbf{y})\\
G_\alpha(z) &= \frac{\exp(ik_\alpha z)}{4\pi z} \quad \text{(fund. sol. Helmholtz eqn., } \alpha = P,S\text{)}
\end{aligned}
$$

- Extension of FMM for Helmholtz to elastodynamics

Differences to define a FMM for **half-space** elastodynamics F.S.

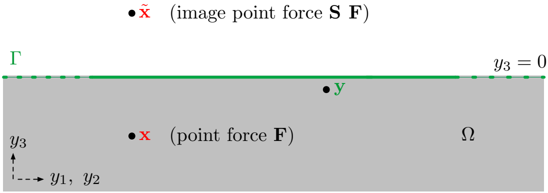
- No multipole expansion of the F.S. currently available
- Need to derive half-space F.S. in form that enables fast computat.
- Need to formulate a new FMM for the elastic half-space F.S.

Elastic Half-space Fundamental solutions

Decomposition of the F.S. into three parts

$$\boldsymbol{U}^{HS}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{U}_\infty(\boldsymbol{x},\boldsymbol{y}) + \bar{\boldsymbol{U}}_\infty(\boldsymbol{x},\boldsymbol{y}) + \boldsymbol{U}_\mathrm{C}(\boldsymbol{x},\boldsymbol{y})$$

- $\boldsymbol{U}_\infty$: elastic full-space F.S.
- $\bar{\boldsymbol{U}}_\infty$: image elastic full-space F.S. (image point force applied at the mirror image source point)



- $\boldsymbol{U}_\mathrm{C}$: complementary F.S.

⇒ Contrary to the scalar case, the method of images allows to only partially satisfy the traction-free condition on the free-surface

Decomposition of the operator

Induced decomposition of integral operators (e.g. single-layer operator):

$$\boldsymbol{v}(\boldsymbol{x}) = \int_{\Gamma_1} \left(\boldsymbol{U}^{HS}(\boldsymbol{x},\boldsymbol{y})\right)^{\mathsf{T}} \boldsymbol{t}(\boldsymbol{y})\, \mathrm{d}S_y, \qquad \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}_\infty(\boldsymbol{x}) + \bar{\boldsymbol{v}}_\infty(\boldsymbol{x}) + \boldsymbol{v}_{\mathrm{C}}(\boldsymbol{x})$$

Fast computation of the three contributions to $\boldsymbol{v}(\boldsymbol{x})$

- Evaluation of $\boldsymbol{v}_\infty(\boldsymbol{x})$ and $\bar{\boldsymbol{v}}_\infty(\boldsymbol{x})$: "standard" FMM (diagonal form-based decomposition of full-space F.S.[4])
- Remaining difficulty: compute the contribution $\boldsymbol{v}_{\mathrm{C}}(\boldsymbol{x})$ involving the complementary F.S.

Goals

- Formulate part $\boldsymbol{U}_{\mathrm{C}}$ of F.S. in a form which enables fast computat.
- Define a fast algorithm to compute the contribution $\boldsymbol{v}_{\mathrm{C}}(\boldsymbol{x})$

[4] [Chaillat *et al.*, CMAME, 2008]

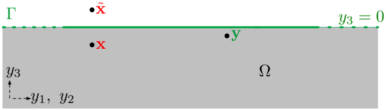
Formulation of half-space F.S. in Fourier space

General idea:

- Fourier transform against first 2 coordinates $(r_1, r_2, y_3) \to (\xi_1, \xi_2, y_3)$
- Classical result for Helmholtz equation[5], for $y_3 > 0$

$$G(r) = \frac{exp(ikr)}{4\pi r} = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{-y_3\sqrt{\xi^2-k^2}} e^{i\xi\left(r_1\cos\alpha + r_2\sin\alpha\right)} \frac{\xi}{\sqrt{\xi^2-k^2}} d\xi d\alpha$$

Methodology:



1. Full-space F.S.: Derive stress vector $\hat{\boldsymbol{p}}_\infty(y_3 = 0)$
2. Image full-space F.S.: Derive stress vector $\hat{\bar{\boldsymbol{p}}}_\infty(y_3 = 0)$
3. Complementary term: Boundary condition

$$\hat{\boldsymbol{p}}_\infty + \hat{\bar{\boldsymbol{p}}}_\infty + \hat{\boldsymbol{p}}_C = 0 \qquad (y_3 = 0)$$

4. Obtain F.S. $\hat{\boldsymbol{U}}_C$ in Fourier space

[5] [Greengard *et al.*, IEEE Comp. Science. Engrg, 1998]

Complementary part $\boldsymbol{U}_{\mathrm{C}}$

Expression in Fourier space

$$\hat{\boldsymbol{U}}_{\mathrm{C}}(\xi_1,\xi_2,y_3,x_3) = \frac{s_{\mathrm{P}}}{4\pi^2\delta(\xi)\mu k_{\mathrm{S}}^2}\left(2\boldsymbol{V}_{\mathrm{S}}(\xi_1,\xi_2)e^{s_{\mathrm{S}}y_3}+(k_{\mathrm{S}}^2-2\xi^2)\boldsymbol{V}_{\mathrm{P}}(\xi_1,\xi_2)e^{s_{\mathrm{P}}y_3}\right)^{\mathrm{T}}$$
$$\times\left(2\boldsymbol{V}'_{\mathrm{S}}(\xi_1,\xi_2)e^{s_{\mathrm{S}}x_3}+(k_{\mathrm{S}}^2-2\xi^2)\boldsymbol{V}'_{\mathrm{P}}(\xi_1,\xi_2)e^{s_{\mathrm{P}}x_3}\right)$$

$$\boldsymbol{V}_{\mathrm{S}} = \left\{\mathrm{i}\xi_1 s_{\mathrm{S}} \quad \mathrm{i}\xi_2 s_{\mathrm{S}} \quad \xi^2\right\}, \qquad \boldsymbol{V}_{\mathrm{P}} = \left\{\mathrm{i}\xi_1/s_{\mathrm{P}} \quad \mathrm{i}\xi_2/s_{\mathrm{P}} \quad 1\right\}$$

$$s_\alpha = \sqrt{\xi^2 - k_\alpha^2}, \qquad \xi^2 = \xi_1^2 + \xi_2^2$$

Complementary part $\boldsymbol{U}_{\mathrm{C}}$

Expression in Fourier space

$$\hat{\boldsymbol{U}}_{\mathrm{C}}(\xi_1,\xi_2,y_3,x_3) = \frac{s_{\mathrm{P}}}{4\pi^2\delta(\xi)\mu k_{\mathrm{S}}^2}\left(2\boldsymbol{V}_{\mathrm{S}}(\xi_1,\xi_2)e^{s_{\mathrm{S}}y_3}+(k_{\mathrm{S}}^2-2\xi^2)\boldsymbol{V}_{\mathrm{P}}(\xi_1,\xi_2)e^{s_{\mathrm{P}}y_3}\right)^{\mathrm{T}}$$
$$\times\left(2\boldsymbol{V}'_{\mathrm{S}}(\xi_1,\xi_2)e^{s_{\mathrm{S}}x_3}+(k_{\mathrm{S}}^2-2\xi^2)\boldsymbol{V}'_{\mathrm{P}}(\xi_1,\xi_2)e^{s_{\mathrm{P}}x_3}\right)$$

$$\boldsymbol{V}'_{\mathrm{S}} = \left\{\mathrm{i}\xi_1 s_{\mathrm{S}} \quad \mathrm{i}\xi_2 s_{\mathrm{S}} \quad -\xi^2\right\},\qquad \boldsymbol{V}'_{\mathrm{P}} = \left\{\mathrm{i}\xi_1/s_{\mathrm{P}} \quad \mathrm{i}\xi_2/s_{\mathrm{P}} \quad -1\right\}$$

Complementary part $\boldsymbol{U}_{\mathrm{C}}$

Expression in Fourier space

$$\hat{\boldsymbol{U}}_{\mathrm{C}}(\xi_1,\xi_2,y_3,x_3) = \frac{s_{\mathrm{P}}}{4\pi^2\delta(\xi)\mu k_{\mathrm{S}}^2}\left(2\boldsymbol{V}_{\mathrm{S}}(\xi_1,\xi_2)e^{s_{\mathrm{S}}y_3}+(k_{\mathrm{S}}^2-2\xi^2)\boldsymbol{V}_{\mathrm{P}}(\xi_1,\xi_2)e^{s_{\mathrm{P}}y_3}\right)^{\mathrm{T}}$$
$$\times\left(2\boldsymbol{V}'_{\mathrm{S}}(\xi_1,\xi_2)e^{s_{\mathrm{S}}x_3}+(k_{\mathrm{S}}^2-2\xi^2)\boldsymbol{V}'_{\mathrm{P}}(\xi_1,\xi_2)e^{s_{\mathrm{P}}x_3}\right)$$

$$\delta(\xi) = (k_{\mathrm{S}}^2-2\xi^2)^2-4\xi^2 s_{\mathrm{S}}s_{\mathrm{P}}, \quad s_\alpha = \sqrt{\xi^2-k_\alpha^2}$$

Complementary part $\boldsymbol{U}_{\mathrm{C}}$

Expression in Fourier space

$$\hat{\boldsymbol{U}}_{\mathrm{C}}(\xi_1,\xi_2,y_3,x_3) = \frac{s_{\mathrm{P}}}{4\pi^2\delta(\xi)\mu k_{\mathrm{S}}^2}\left(2\boldsymbol{V}_{\mathrm{S}}(\xi_1,\xi_2)e^{s_{\mathrm{S}}y_3}+(k_{\mathrm{S}}^2-2\xi^2)\boldsymbol{V}_{\mathrm{P}}(\xi_1,\xi_2)e^{s_{\mathrm{P}}y_3}\right)^{\mathrm{T}}$$
$$\times\left(2\boldsymbol{V}'_{\mathrm{S}}(\xi_1,\xi_2)e^{s_{\mathrm{S}}x_3}+(k_{\mathrm{S}}^2-2\xi^2)\boldsymbol{V}'_{\mathrm{P}}(\xi_1,\xi_2)e^{s_{\mathrm{P}}x_3}\right)$$

k_{R} such that $\delta(k_{\mathrm{R}})=0$ is the Rayleigh wavenumber ($k_{\mathrm{P}}<k_{\mathrm{S}}<k_{\mathrm{R}}$)

Complementary part U_C

Expression in Fourier space

$$\hat{U}_C(\xi_1,\xi_2,y_3,x_3) = \frac{s_P}{4\pi^2\delta(\xi)\mu k_S^2}\big(2V_S(\xi_1,\xi_2)e^{s_S y_3} + (k_S^2 - 2\xi^2)V_P(\xi_1,\xi_2)e^{s_P y_3}\big)^T$$
$$\times \big(2V'_S(\xi_1,\xi_2)e^{s_S x_3} + (k_S^2 - 2\xi^2)V'_P(\xi_1,\xi_2)e^{s_P x_3}\big)$$

k_R such that $\delta(k_R) = 0$ is the Rayleigh wavenumber ($k_P < k_S < k_R$)

Expression in physical coordinates

$$U_C(x,y) = \int_0^{+\infty}\int_0^{2\pi} e^{i\xi(\cos\alpha(y_1-x_1)+\sin\alpha(y_2-x_2))}\hat{U}_C(\xi\cos\alpha,\xi\sin\alpha,y_3,x_3)\xi\, d\alpha d\xi$$

FMM-friendly formulation: integrand involves product of a function of x and a function of y $\rightarrow$ essential feature for fast evaluation

Numerical evaluation

$$\mathbf{U}_\mathrm{C}(\mathbf{x},\mathbf{y}) = \int_0^{+\infty} \underbrace{\int_0^{2\pi} e^{\mathrm{i}\xi(\cos\alpha(y_1-x_1)+\sin\alpha(y_2-x_2))}\,\hat{\mathbf{U}}_\mathrm{C}(\xi\cos\alpha,\xi\sin\alpha,y_3,x_3)\xi\,\mathrm{d}\alpha}_{\mathbf{I}_C(\xi,\mathbf{x},\mathbf{y})}\,\mathrm{d}\xi$$

Evaluation in standard BEM (not adapted to FMM)

- Integration over α analytical (yields $J_i(\xi|\mathbf{y}-\mathbf{x}|)$, $0 \leq i \leq 2$)
- Destroys separation of variables $\mathbf{x}$ and $\mathbf{y}$

Numerical integration in context of FMM: double integral in (α,ξ) space

- Quadrature rule over α (inner integral)
  - Smooth, periodic functions
  - $\Rightarrow$ Simple trapezoidal rule
- Quadrature rule over ξ (integrand: $\mathbf{I}_C(\xi,\mathbf{x},\mathbf{y})$)
  - Oscillatory functions: difficult to integrate numerically
  - Singularity due to Rayleigh pole
  - Quadrature for $\mathbf{x}$ and $\mathbf{y}$ lying in a prescribed parameter space
  - $\Rightarrow$ Non-classical quadrature rule necessary

Generalized Gaussian quadrature rules

Generalized Gaussian quadrature rules

⇒ Well adapted to the definition of the FMM

- Definition of a FMM at low frequencies [Greengard *et al.*, 1998]
- Definition of a wideband FMM [Cheng et al. , 2006]

⇒ Construction of quadrature rule is involved and integrand-specific

Generalized Gaussian quadrature rules

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$\Rightarrow$ Construction of quadrature rule is involved and integrand-specific

Generic methodology for constructing GGQ rule [Bremer *et al.*, 2010]

- Output quadrature formula with N_ξ nodes

$$\sum_{j=1}^{N_\xi} \mathbf{I}_C(\xi_j, \boldsymbol{x}, \boldsymbol{y}) w_j \simeq \int_0^{\xi_{max}} \mathbf{I}_C(\xi, \boldsymbol{x}, \boldsymbol{y}) d\xi$$

- Quadrature integrating collection of input functions $\phi_1, \ldots, \phi_m$
  - m square integrable functions
  - Reproduce the singular and oscillatory behaviors of $\mathbf{I}_C$ on $[0, \xi_{max}]$
  - Sampled values of $\boldsymbol{x}$ and $\boldsymbol{y}$ in the prescribed parameter space

  $\Rightarrow$ Large number m of input functions for a good accuracy

- N_ξ is small if the rank of the input functions is small

$\Rightarrow$Developed a version of GGQ custom-tailored to our needs

New Multipole Expansion for elastic half-space

Diagonal form-based decomposition of the complementary term

$$\boldsymbol{U}_{\mathrm{C}}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi^2\mu k_{\mathrm{S}}^2}\sum_{a,b=\mathrm{P,S}}\int_0^{\xi_{max}}\int_0^{2\pi} e^{\mathrm{i}\xi(y_1\cos\alpha+y_2\sin\alpha)}e^{y_3 s_a}\mathcal{O}_{ab}(\xi)\boldsymbol{V}'_b(\xi,\alpha)$$

$$\boldsymbol{V}_a^{\mathrm{T}}(\xi,\alpha)e^{\mathrm{i}\xi(-x_1\cos\alpha-x_2\sin\alpha)}e^{x_3 s_b}\,\mathrm{d}\alpha\mathrm{d}\xi$$

where $\mathcal{O}_{\mathrm{SS}}(\xi) = \dfrac{4\xi s_{\mathrm{P}}}{\delta}, \quad \mathcal{O}_{\mathrm{SP}}(\xi) = \mathcal{O}_{\mathrm{PS}}(\xi) = \dfrac{2\xi\beta s_{\mathrm{P}}}{\delta}, \quad \mathcal{O}_{\mathrm{PP}}(\xi) = \dfrac{\xi\beta^2 s_{\mathrm{P}}}{\delta}.$

Advantages of the Multipole Expansion

- FMM-friendly formulation: separation of $\boldsymbol{x}$ and $\boldsymbol{y}$
- Rank one formulation

New Algorithm for elastic half-space

Evaluation of single-layer potential: $\boldsymbol{v}_C(\boldsymbol{x}) = \int_{\Gamma_1} \boldsymbol{U}_C^T(\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{t}(\boldsymbol{y})\,dS_y$

1. Computation of multipole moments ($a, b = \mathrm{P}, \mathrm{S}$)

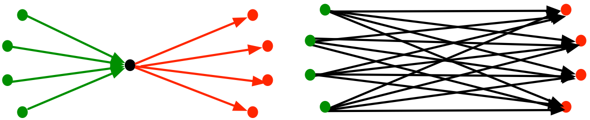
$$\mathcal{R}_{ab}(\xi,\alpha) = \int_{\Gamma_1} e^{i\xi(y_1\cos\alpha + y_2\sin\alpha)} e^{y_3 s_a}\,\mathcal{O}_{ab}(\xi)\boldsymbol{V}'_b(\xi,\alpha)\;\boldsymbol{t}(\boldsymbol{y})\,dS_y$$

2. Evaluation at observation points (quadrature in Fourier space)

$$\boldsymbol{v}_C^k(\boldsymbol{x}) = \sum_{\ell=1}^{N_\xi}\sum_{j=1}^{N_\alpha} w_j^\alpha w_\ell^\xi e^{i\xi_\ell(-x_1\cos\alpha_j - x_2\sin\alpha_j)} \sum_{a,b=\mathrm{P},\mathrm{S}} \boldsymbol{V}_a^k(\xi_\ell,\alpha_j)e^{x_3 s_b}\,\mathcal{R}_{ab}(\xi_\ell,\alpha_j)$$

FM-BEM

BEM

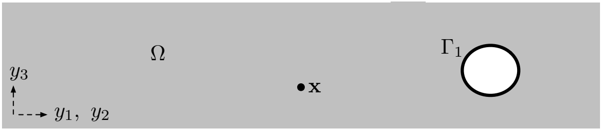


Studied case

Current assumption: Γ_1 does not intercept the free-surface

- Multipole Expansion for $\boldsymbol{U}_\mathrm{C}$ always valid
- Otherwise: Singularities if $\boldsymbol{x} = \boldsymbol{y}$ on the free-surface (requires additional work)

$$\int_0^{\xi_{max}} \int_0^{2\pi} e^{i\xi(y_1 \cos\alpha + y_2 \sin\alpha)} e^{y_3 s_a} \mathcal{O}_{ab}(\xi) \boldsymbol{V}'_b(\xi,\alpha) \boldsymbol{V}_a^\mathrm{T}(\xi,\alpha) e^{i\xi(-x_1 \cos\alpha - x_2 \sin\alpha)} e^{x_3 s_b} \, d\alpha d\xi$$



Complexity Estimate: computation of $\boldsymbol{U}_\mathrm{C}$

- N: number of BE DOFs
- N_α, N_ξ: number of quadrature points inner and outer integrals

- For each quadrature point and each DOF
- Evaluating $\boldsymbol{v}_\mathrm{C}$ costs $O(N_\xi N_\alpha N)$

$\Rightarrow$Numerical tests to determine complexities of N_α and N_ξ with N

Numerical experiments

1. Motivations and background
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Studied configurations

Verification of the new multipole expansion in terms of

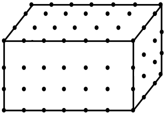
- Accuracy
- Complexity of the matrix-vector product

$$\sum_{j}^{N} \boldsymbol{U}_{\mathrm{C}}(\boldsymbol{x}_i, \boldsymbol{y}_j)\mathbf{t}(\boldsymbol{y}_j) \quad \text{(for given sets of points } x_i, y_j \quad 1 \le i, j \le N)$$

Discretization: Fixed density of points per S-wavelength

$\Rightarrow$ The circular frequency ω is increased to increase $\#$ of DOFs

Rectangular domain: (non-dimensional spatial coordinates)



$$-4 \le y_1, y_2 \le 4, \quad 1 \le y_3 \le 4$$

Mechanical properties: $\mu = 3, \quad \nu = 0.25, \quad \rho = 1$

$$c_{\mathrm{S}} = \sqrt{\frac{\mu}{\rho}}, \quad c_{\mathrm{P}} = c_{\mathrm{S}}\sqrt{\frac{2 - 2\nu}{1 - 2\nu}}, \quad k_{\mathrm{P}} = \frac{\omega}{c_{\mathrm{P}}}, \quad k_{\mathrm{S}} = \frac{\omega}{c_{\mathrm{S}}}$$

Accuracy of the multipole expansions

Comparisons with the code provided by B.B. Guzina[6] (F.S. of the layered elastic half-space)

$$\varepsilon_{ij} = \frac{||\boldsymbol{U}_{ij}^{HS} - (\boldsymbol{U}_{ij}^{HS})_{guz.}||_2}{||(\boldsymbol{U}_{ij}^{HS})_{guz.}||_2}$$

(if inner integral very accurate)

Accuracy w.r.t N_ξ (fixed $\omega = 1$)

N_ξ	ε_{11}	ε_{12}	ε_{13}	ε_{31}	ε_{33}
12	$4.1\ 10^{-2}$	$1.9\ 10^{-2}$	$1.5\ 10^{-1}$	$2.1\ 10^{-1}$	$2.7\ 10^{-2}$
15	$4.4\ 10^{-3}$	$3.1\ 10^{-3}$	$2.3\ 10^{-2}$	$4.8\ 10^{-2}$	$1.8\ 10^{-2}$
16	$1.9\ 10^{-3}$	$6.3\ 10^{-4}$	$4.8\ 10^{-3}$	$6.3\ 10^{-3}$	$1.5\ 10^{-3}$
17	$1.2\ 10^{-3}$	$4.2\ 10^{-4}$	$3.2\ 10^{-3}$	$4.5\ 10^{-3}$	$1.4\ 10^{-3}$
23	$5.5\ 10^{-5}$	$1.9\ 10^{-4}$	$1.7\ 10^{-4}$	$1.6\ 10^{-4}$	$1.8\ 10^{-4}$

$\Rightarrow$ Better accuracy if $\#$ points in the outer integral increases

[6] [B.B. Guzina and Pak, QJMAM, 2001])

Trapezoidal Quadrature Rule

$$\boldsymbol{U}_C(\boldsymbol{x},\boldsymbol{y}) = \int_0^{\xi_{max}} \underbrace{\int_0^{2\pi} e^{i\xi(\cos\alpha(y_1-x_1)+\sin\alpha(y_2-x_2))} \hat{\boldsymbol{U}}_C(\xi\cos\alpha, \xi\sin\alpha, y_3, x_3) d\alpha}_{\mathbf{I}_C(\xi,\boldsymbol{x},\boldsymbol{y})} \; \xi \, d\xi$$

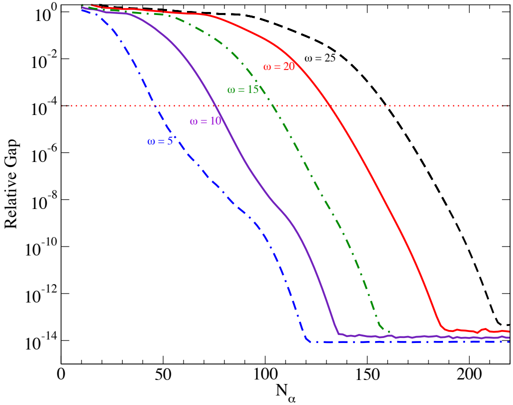
Choice of N_α? depends on Rayleigh wavenumber k_r

$$\xi_{max} = 2k_r + \frac{10}{|\min(y_3)|}; \qquad \forall \xi \geq \xi_{max} \qquad |\mathbf{I}_C(\xi,\boldsymbol{x},\boldsymbol{y})| \ll 10^{-6}$$

Trapezoidal Quadrature Rule

$$\boldsymbol{U}_{\mathrm{C}}(\boldsymbol{x},\boldsymbol{y}) = \int_0^{\xi_{max}} \underbrace{\int_0^{2\pi} e^{\mathrm{i}\xi(\cos\alpha(y_1-x_1)+\sin\alpha(y_2-x_2))}\hat{\boldsymbol{U}}_{\mathrm{C}}(\xi\cos\alpha,\xi\sin\alpha,y_3,x_3)\mathrm{d}\alpha\ \xi}_{\mathbf{I}_C(\xi,\boldsymbol{x},\boldsymbol{y})}\ \mathrm{d}\xi$$

Choice of N_α? depends on circular frequency ω

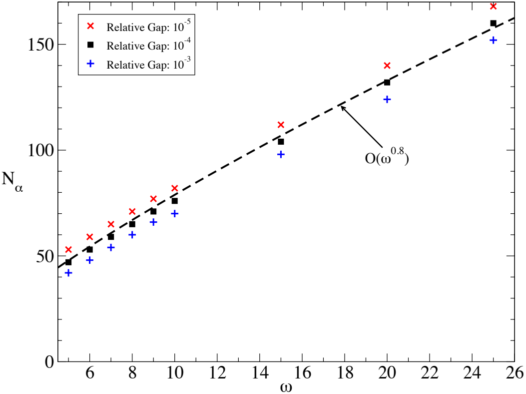


Relative Discrepancy on $\sum_j \boldsymbol{U}_{\mathrm{C}}(\boldsymbol{x}_i,\boldsymbol{y}_j)\mathbf{t}(\boldsymbol{y}_j)$ with a reference solution ($N_\alpha = 250$)

Trapezoidal Quadrature Rule

$$\boldsymbol{U}_{\mathrm{C}}(\boldsymbol{x},\boldsymbol{y}) = \int_0^{\xi_{max}} \underbrace{\int_0^{2\pi} e^{\mathrm{i}\xi(\cos\alpha(y_1-x_1)+\sin\alpha(y_2-x_2))}\hat{\boldsymbol{U}}_{\mathrm{C}}(\xi\cos\alpha,\xi\sin\alpha,y_3,x_3)\mathrm{d}\alpha}_{\mathbf{I}_C(\xi,\boldsymbol{x},\boldsymbol{y})}\ \xi\,\mathrm{d}\xi$$

Selection rule for N_α



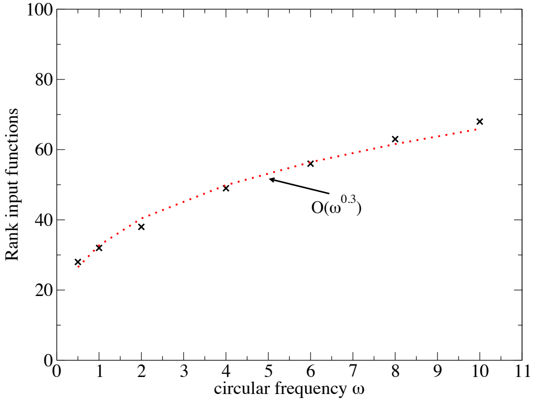
$$O(\omega) = O(N^{1/2})$$

$$\Rightarrow N_\alpha = O(N^{0.4})$$

Generalized Gaussian Quadrature

$$\boldsymbol{U}_{\mathrm{C}}(\boldsymbol{x},\boldsymbol{y}) = \int_0^{\xi_{max}} \sum_{j=1}^{N_\alpha} w_j^\alpha e^{\mathrm{i}\xi(\cos\alpha_j(y_1-x_1)+\sin\alpha_j(y_2-x_2))} \hat{\boldsymbol{U}}_{\mathrm{C}}(\xi\cos\alpha_j, \xi\sin\alpha_j, y_3, x_3)\, \xi\, \mathrm{d}\xi$$

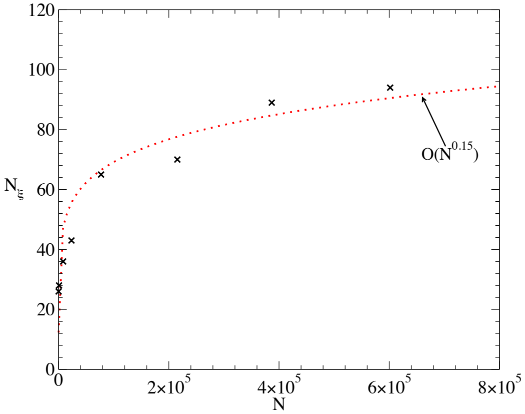
Rank of the input functions



Rank of input funct. is $O(\omega^{0.3})$

$\Rightarrow N_\xi$ is expected to be of order $N_\xi = O(\omega^{0.3}) = O(N^{0.15})$

Complexity of the number of points with N

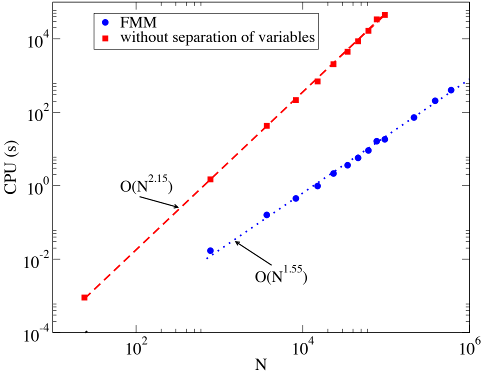


$\Rightarrow$ Custom-tailored algorithm leads to $N_\xi = O(N^{0.15})$

$\Rightarrow$ The anticipated trend is verified a posteriori

Numerical Complexity Check

Theoretical complexity estimates to compute $\boldsymbol{v}_C$



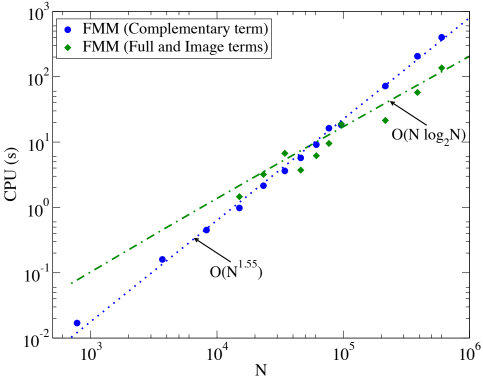
$$N_\xi = O(N^{0.15})$$
$$N_\alpha = O(N^{0.4})$$

- Without separation of variables $\boldsymbol{x}$, $\boldsymbol{y}$: $O(N_\xi N^2) \Rightarrow O(N^{2.15})$
- FMM: $O(N_\alpha N_\xi N) \Rightarrow O(N^{1.55})$

$\Rightarrow$ Good agreement between numerical and theoretical complexity

Numerical Complexity Check

Theoretical complexity estimate to compute $\boldsymbol{v} = \boldsymbol{v}_\infty + \bar{\boldsymbol{v}}_\infty + \boldsymbol{v}_C$

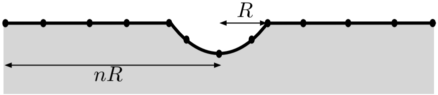
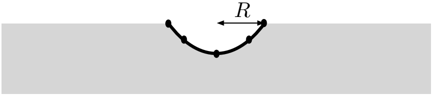


- "New" FMM to compute $\boldsymbol{v}_C$: $O(N^{1.55})$
- "Standard" FMM to compute sum of $\boldsymbol{v}_\infty$ & $\bar{\boldsymbol{v}}_\infty$: $O(N \log N)$

$\Rightarrow$ Similar order of magnitude to compute the various terms

Comparison with computation full-space terms

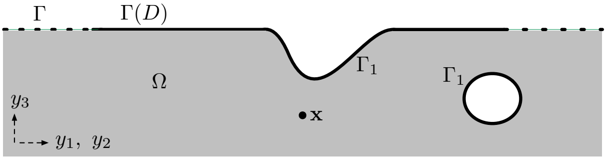
Comparison of the number of DOFs: diffraction of a plane wave by a semi-spherical canyon

Full-space F.S.	Half-space F.S.
	
Artificial truncation of the free-surface N_F $A_F = \pi R^2(n^2+1)$	No truncation $N_{H.S.}$ $A_{H.S.} = 2\pi R^2$

- Due to geometrical considerations $N_F = \frac{n^2+1}{2} N_{H.S.}$
- If $n = 5$ (usual used value) $N_F = 13\ N_{H.S.}$

Summary and conclusions

- Formulation of the multipole expansions for the half-space elastodynamic fundamental solutions
- Tailored algorithm to compute Generalized Gaussian Quadrature
- New FMM proposed for problems in an elastic half-space
- Accuracy of multipole expansion tested against published results
- Complexity of a matrix-vector product checked numerically



Future work: half-space formulation

- Implementation of complete FMM for the elastic half-space F.S.
  - Efficient way to compute the image term?
  - Singularities when ***x*** and ***y*** are both on the free surface?

- Comparisons of accuracy and numerical efficiency with classical formulation based on elastic full-space F.S.
  - Can we run higher frequencies than with previous formulation?
  - Influence of truncation of the free-surface for full-space formulation?

- Parallelization to run even higher frequencies

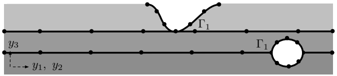
- Preconditioning: algebraic preconditioners are only moderately efficient

Future work: layered half-space formulation

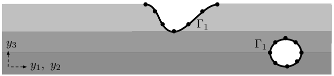
- Current approach with homogeneous F.S.:
  - Boundary Element-Boundary Element coupling
  - Mesh all interfaces
  - Conforming elements because strong coupling

- Usefulness of a FMM for the layered visco-elastic half-space F.S. (to be formulated)
  - Mesh drastically reduced
  - Not straightforward but a similar approach should be applicable

Homogeneous F.S.

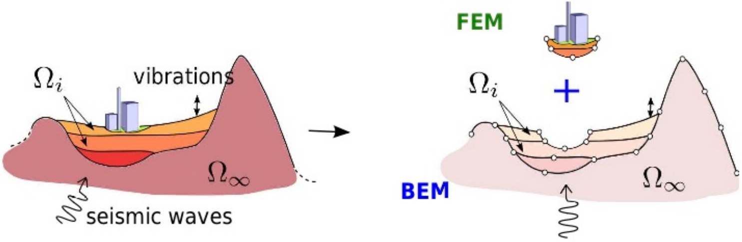


Layered Half-space F.S.



Future work: FEM/BEM coupling

Exploits complementarity between BEM and FEM



- Finite Element Method
  - Near field
  - Bounded domains with complex geometries, heterogeneities
- Fast Multipole Accelerated Boundary Element Method
  - Far field
  - Semi-infinite linear and piecewise homogeneous domains

Future work: FEM/BEM coupling

Non-reflecting boundary conditions

- High-order Absorbing boundary conditions
- Perfectly Matched Layer
- FEM/BEM coupling

Coupling FEM/FM-BEM

- Iterative DDM-like algorithm
- BEM influence matrix not assembled

Coupling with full-space formulation: PhD thesis Eva Grasso (2012)

- Sequential approach: easy to implement (black box) but slow
  - Alternate solution in Ω_F and Ω_B
  - Update boundary conditions on the interface until convergence
- Simultaneous approach: fast but involved implementation
  - Build a global system of equations
  - Simultaneous solution of the global system

Coupling with half-space formulation

- Promising
- But involved implementation if simultaneous approach used

Future work: time-domain problems

- Frequency-domain computations at sampling frequencies and Fourier transform
  - Depends strongly on a proper choice of parameters
- Use of the time-domain F.S.
  - FMM for full-space F.S. is already involved
- Other Possible approach: Convolution Quadrature Method
  - More stable time stepping procedure
  - Already used for visco-elastodynamics BEM accelerated by ACA (group of M. Schanz, T.U. Graz)

Future work: Main possible applications

- Soil-structure interaction
- Seismic wave propagation
- Site-effects
- Traffic- or excavation-induced vibrations

Méthode d'équation intégrale accélérée par la méthode multipôle rapide pour les problèmes tridimensionnels de propagation d'ondes élastiques dans un demi-espace

Stéphanie Chaillat

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Séminaire LaMSID
Mardi 4 Septembre 2012