A model for crack propagation based on viscous approximation

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Séminaire LaMSID, October 12th, 2010

Joint work with Rodica Toader (Udine)

Aim of the work

- Crack propagation: We want to determine the evolution of a brittle body

  Griffith's principle: The crack's energy is proportional to its measure

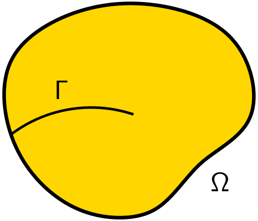
- Viscous approximation: Standard method for rate-independent processes

  At every instant the configuration is a stationary point for the total energy

- Differently from previous existence results based on local minimization,

  we give no a-priori prescriptions on the path followed by the crack

Crack growth in brittle materials



$\Omega \subseteq \mathbb{R}^2$ section of an elastic body

$\Gamma \subseteq \Omega$ crack of measure $\ell(\Gamma) < +\infty$

$u : \Omega \setminus \Gamma \to \mathbb{R}$ displacement of the elastic part

Antiplane setting: $(x_1, x_2, x_3) \mapsto (x_1, x_2, u(x_1, x_2) + x_3)$

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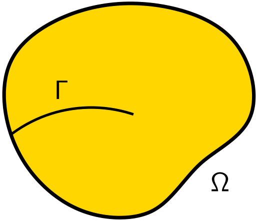

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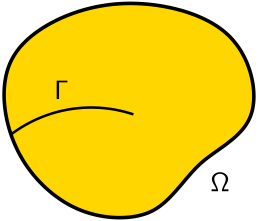

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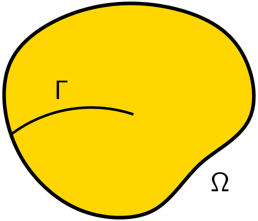

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Find $t \mapsto (u(t), \Gamma(t))$ such that $(u(t), \Gamma(t))$ is an equilibrium for $\mathcal{E}$ taking into account the irreversibility of fracture

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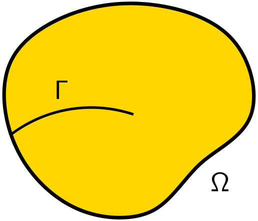

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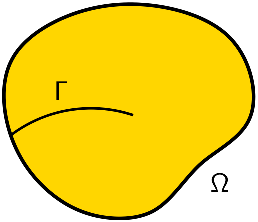

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$$\begin{aligned} \text{Bulk energy} \quad & \mathcal{W}(t;\Gamma) && := \frac{1}{2} \int_{\Omega\setminus\Gamma} |\nabla u(t;\Gamma)|^2 \, dx \\ \text{Crack energy} \quad & \ell(\Gamma) && := \text{length of } \Gamma \\ \text{Total energy} \quad & \mathcal{E}(t;\Gamma) && := \mathcal{W}(t;\Gamma) + \ell(\Gamma) \end{aligned}$$

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Difficulty

The notion of critical point is not defined in this case.

There are two main possible approaches:

- Restrict to global minima instead of all stationary points
- Give some prescriptions on the crack path allowing for local notions

The Francfort-Marigo model

Assumption: For every $t \in [0, T]$ the crack $\Gamma(t)$ is a unilateral global minimizer

$$\mathcal{E}(t; \Gamma(t)) \leq \mathcal{E}(t; \Gamma) \quad \forall \Gamma \supseteq \Gamma(t)$$

Variational model: Francfort-Marigo (1998)

Advantages:

- This notion is well defined for a wide class of cracks (closed or rectifiable)
- Crack path determined by the energy criterion, not prescribed a priori

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- Dal Maso-Toader (2002): Antiplane elasticity
- Chambolle (2003): Planar elasticity
- Francfort-Larsen (2003): Rectifiable cracks, *SBV* deformations
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Drawback: Jumps between different energy wells (because of nonconvexity)

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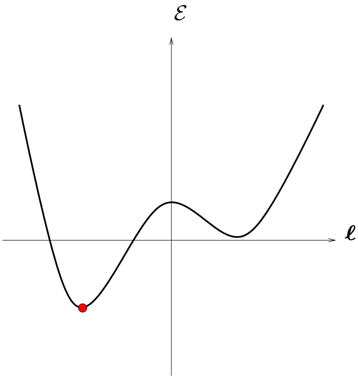
Global minimizers $\Rightarrow$ Jumps between energy wells, overtaking barriers



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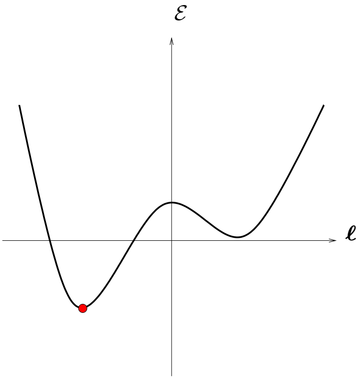

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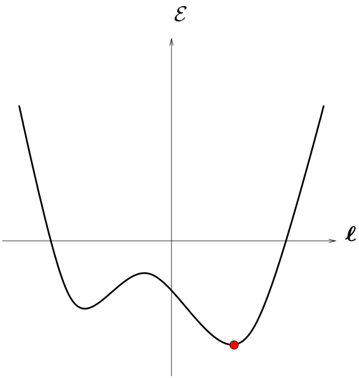

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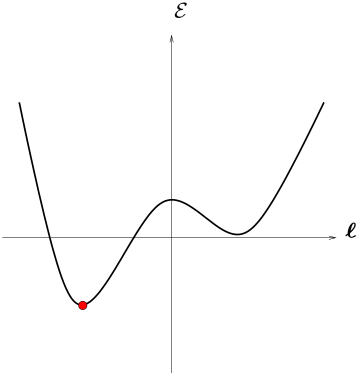

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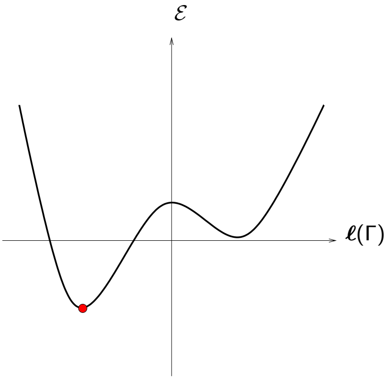
Alternative: Allow just for overtaking "small" energy barriers (Larsen, 2010)

Local minimization

Local minimization $\Rightarrow$ Jumps happen later, without overtaking barriers



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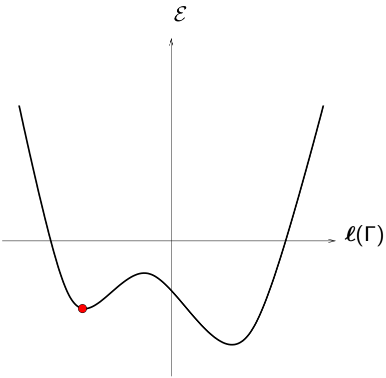

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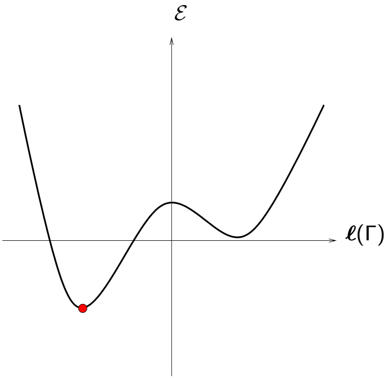

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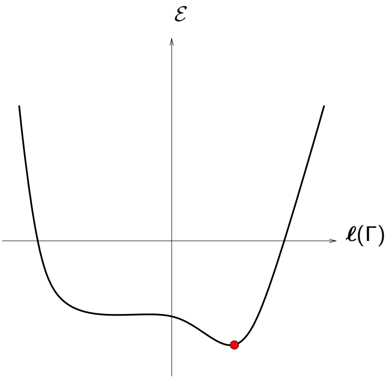

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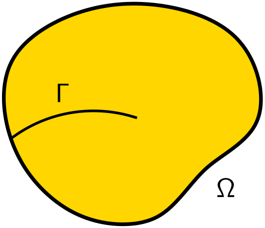

Example: Crack evolving on a fixed curve Γ

(e.g., Γ a straight line, a smooth curve, . . .)

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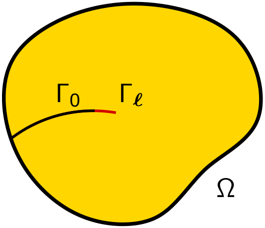

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Γ_ℓ extension of Γ_0 along Γ

$\Rightarrow$ The system depends on only one variable: the length ℓ of the curve

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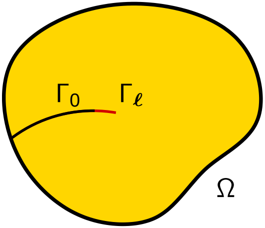

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Case of prescribed crack, via vanishing viscosity

Toader-Zanini (2009), Negri-Ortner (2008), Knees-Mielke-Zanini (2008-2009)

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Goal

- crack path not prescribed (but quite regular)
- allow for local minimization via vanishing viscosity

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Aim: Define a derivative "$\partial_\Gamma \mathcal{W}(t; \Gamma)$"

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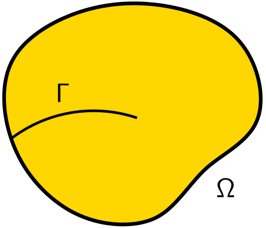

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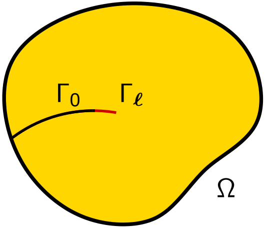

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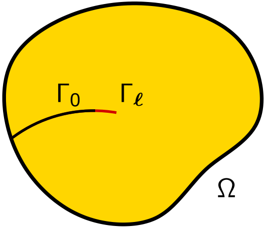

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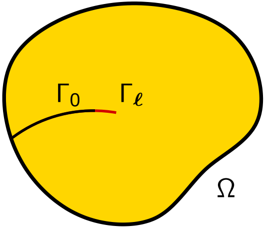

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Fact: The energy release rate is actually independent of the continuation of Γ_0

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Total energy: $\mathcal{E}(t; \Gamma) := \mathcal{W}(t; \Gamma) + \ell(\Gamma)$

Equivalent statement of Griffith's criterion

- Irreversibility: $\dot{\ell}(t) \geq 0$
- Unilateral stability: $\ell(t)$ is stationary for $\mathcal{E}(t; \Gamma)$ among all Γ: $\ell(\Gamma) \geq \ell(t)$
- Energy balance: $\dfrac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{E}(t; \Gamma(t))$ equals the power of the surface forces

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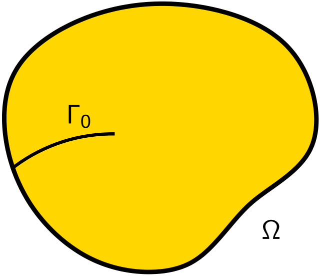
5. Time-continuous limit as $n \to \infty$ with ε fixed: obtain viscous solutions
6. Inviscid limit as $\varepsilon \to 0$: obtain a rate-independent evolution
7. Griffith's criterion in the limit (using some continuity of $G(t, \cdot)$)

For the limit: select a suitable class of admissible cracks

Admissible cracks



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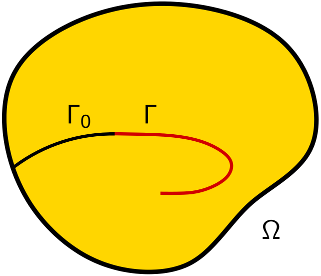

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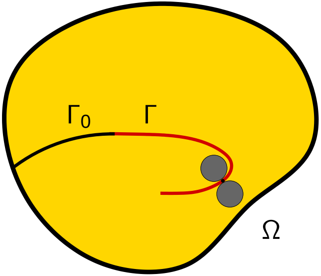

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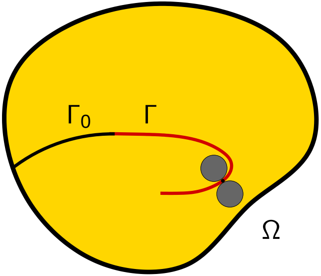

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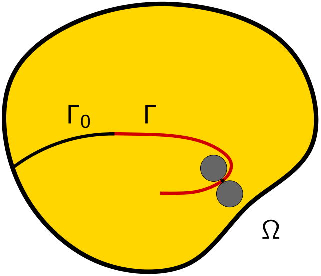
- The curvature is controlled by $\frac{1}{\eta}$

Admissible cracks



Initial crack: Γ_0 simple curve, not tangent to $\partial\Omega$

Admissible cracks: $\Gamma \supseteq \Gamma_0$ of class $C^{1,1}$
(without knowing a priori the crack path)

Condition: The two tangent disks of radius $\eta > 0$
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- In the limit, the tip does not meet the curve Γ, nor the boundary of Ω

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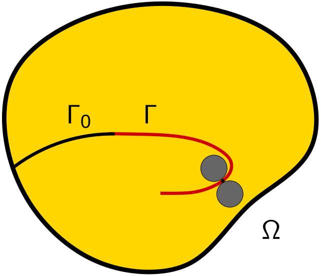

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- It is possible to find limits of class $C^{1,1}$ in $\mathcal{R}_\eta$
- In the limit, the tip does not meet the curve Γ, nor the boundary of Ω
- We have convergence of parametrizations, tangents, and lengths
- We have convergence of the energies and of the energy release rates

Approximation

Time discretization: $(t_{n,0},\, t_{n,1}, \dots, t_{n,n-1},\, t_{n,n})$ such that

$$0 = t_{n,0} < t_{n,1} < \dots < t_{n,n} = T \quad \text{and} \quad \max_{1 \le i \le n} (t_{n,i} - t_{n,i-1}) = 0 \ \text{ as } \ n \longrightarrow \infty$$

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Approximate solutions

- $\Gamma_{\varepsilon,n,0} := \Gamma_0$
- $\Gamma_{\varepsilon,n,i}$ a solution to

$$\min_{\Gamma_{\varepsilon,n,i-1} \subseteq \Gamma \in \mathcal{R}_\eta} \left\{ \mathcal{W}(t_{n,i}; \Gamma) + \ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1}) + \frac{\varepsilon}{2} \frac{\ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}} \right\}$$

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1. Time-continuous limit: $\varepsilon > 0,\;\; n \to \infty$

   Then $\Gamma_{\varepsilon,n,i}$ converges to a continuous function $t \mapsto \Gamma_\varepsilon(t)$

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2. Rate-independent limit: $\varepsilon \to 0$

   Then $\Gamma_\varepsilon(t)$ converges to a function $t \mapsto \Gamma(t)$

Approximation

Time discretization: $(t_{n,0},\ t_{n,1}, \dots, t_{n,n-1},\ t_{n,n})$ such that

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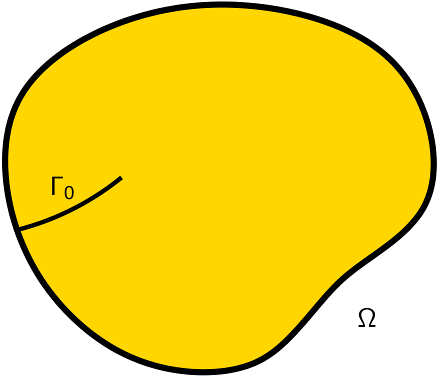
   Then $\Gamma_{\varepsilon,n,i}$ converges to a continuous function $t \mapsto \Gamma_\varepsilon(t)$

2. Rate-independent limit: $\varepsilon \to 0$

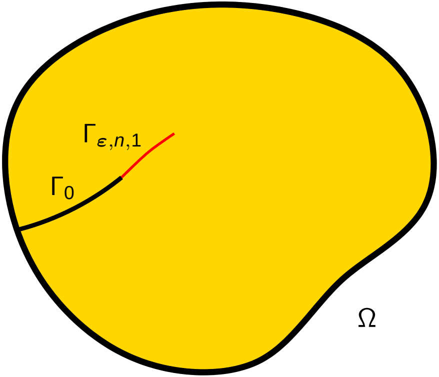
   Then $\Gamma_\varepsilon(t)$ converges to a function $t \mapsto \Gamma(t)$ with jumps

Passage to the limit



Start with the initial condition

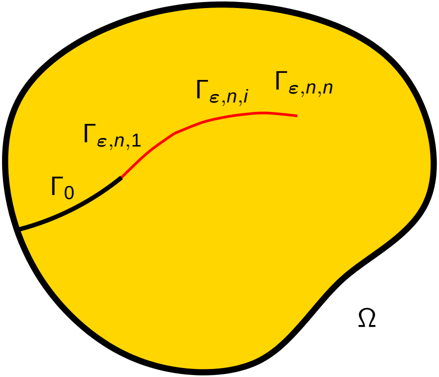
Passage to the limit



Solve inductively the viscous problems in $\mathcal{R}_{\boldsymbol{\eta}}$

$$\Gamma_{\varepsilon,n,i} \in \mathcal{R}_{\boldsymbol{\eta}}$$

Passage to the limit

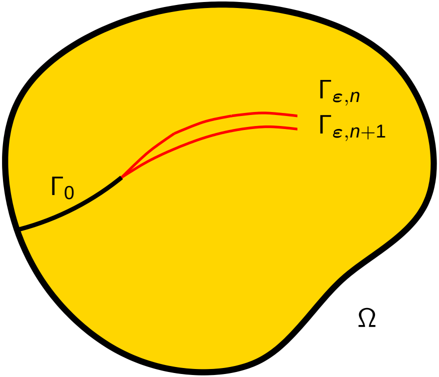


Solve inductively the viscous problems in $\mathcal{R}_{\boldsymbol{\eta}}$

$$\Gamma_{\varepsilon,n,i} \in \mathcal{R}_{\boldsymbol{\eta}}$$

Consider the piecewise constant interpolations $\Gamma_{\varepsilon,n}(t)$

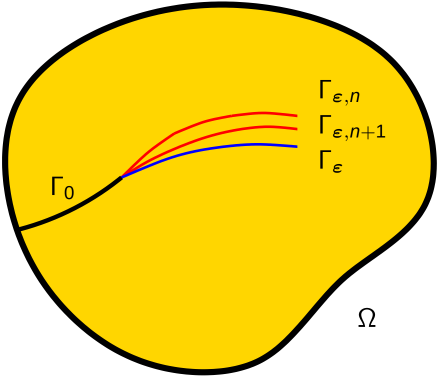
Passage to the limit



Obtain an approximate sequence

$$\Gamma_{\varepsilon,n}(t) \in \mathcal{R}_{\boldsymbol{\eta}}$$

Passage to the limit

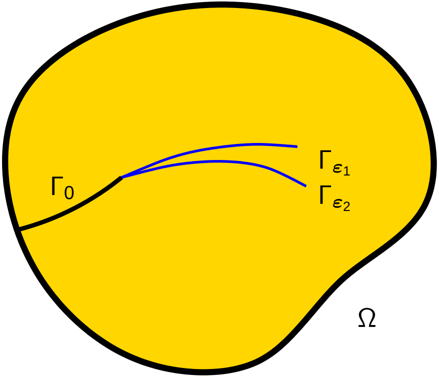


Time-continuous limit at ε fixed

$$\Gamma_{\varepsilon,n}(t) \longrightarrow \Gamma_{\varepsilon}(t) \quad \text{in } \mathcal{R}_{\eta}$$

$$\mathcal{E}(t;\Gamma_{\varepsilon,n}(t)) \longrightarrow \mathcal{E}(t;\Gamma_{\varepsilon}(t))\,, \quad G\,(t;\Gamma_{\varepsilon,n}(t)) \longrightarrow G\,(t;\Gamma_{\varepsilon}(t))$$

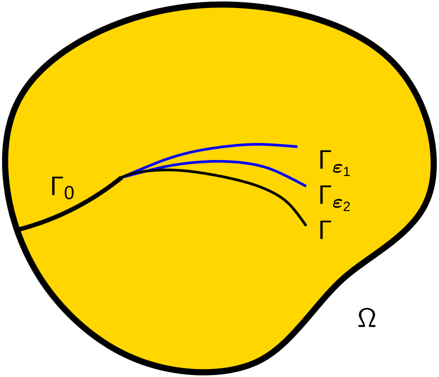
Passage to the limit



Obtain a sequence of viscous solutions

$$\Gamma_\varepsilon(t) \in \mathcal{R}_\eta$$

Passage to the limit



Rate-independent limit

$$\Gamma_\varepsilon(t) \longrightarrow \Gamma(t) \quad \text{in } \mathcal{R}_\eta$$

$$\mathcal{E}(t;\Gamma_\varepsilon(t)) \longrightarrow \mathcal{E}(t;\Gamma(t)), \quad G(t;\Gamma_\varepsilon(t)) \longrightarrow G(t;\Gamma(t))$$

Griffith's criterion

- $\Gamma_{\varepsilon,n,i}$ minimizes $\mathcal{W}(t_{n,i};\Gamma) + \ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1}) + \frac{\varepsilon}{2}\frac{\ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}}$

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- Passing to the derivatives

$$-G\Big(t;\Gamma_{\varepsilon,n}(t)\Big) \quad + \quad 1 \quad + \quad \varepsilon\,\dot{\ell}_{\varepsilon,n}(t) \quad \geq 0$$

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- We have the equality when $\dot{\ell}_{\varepsilon,n}(t) > 0$

Discrete Griffith's criterion

- $\dot{\ell}_{\varepsilon,n}(t) \geq 0$
- $1 - G\Big(t;\Gamma_{\varepsilon,n}(t)\Big) + \varepsilon\,\dot{\ell}_{\varepsilon,n}(t) \geq 0$
- $\dot{\ell}_{\varepsilon,n}(t)\,\Big(1 - G\Big(t;\Gamma_{\varepsilon,n}(t)\Big) + \varepsilon\,\dot{\ell}_{\varepsilon,n}(t)\Big) = 0$

Griffith's criterion

- $\Gamma_{\varepsilon,n,i}$ minimizes $\mathcal{W}(t_{n,i};\Gamma) + \ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1}) + \frac{\varepsilon}{2}\frac{\ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}}$
- Passing to the derivatives

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- We have the equality when $\dot{\ell}_{\varepsilon,n}(t) > 0$
- Then we pass to the limit

Viscous Griffith's criterion

- $\dot{\ell}_{\varepsilon}(t) \geq 0$
- $1 - G\Big(t;\Gamma_{\varepsilon}(t)\Big) + \varepsilon\,\dot{\ell}_{\varepsilon}(t) \geq 0$
- $\dot{\ell}_{\varepsilon}(t)\Big(1 - G\Big(t;\Gamma_{\varepsilon}(t)\Big) + \varepsilon\,\dot{\ell}_{\varepsilon}(t)\Big) = 0$

Griffith's criterion

- $\Gamma_{\varepsilon,n,i}$ minimizes $\mathcal{W}(t_{n,i};\Gamma) + \ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1}) + \frac{\varepsilon}{2}\frac{\ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}}$

- Passing to the derivatives

$$-G\Big(t;\Gamma_{\varepsilon,n}(t)\Big) \quad + \quad 1 \quad + \quad \varepsilon\,\dot{\ell}_{\varepsilon,n}(t) \quad \geq 0$$

- We have the equality when $\dot{\ell}_{\varepsilon,n}(t) > 0$

- Then we pass to the limit

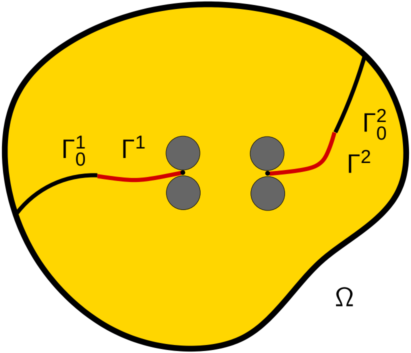
Griffith's criterion

- $\dot{\ell}(t) \geq 0$

- $1 - G\Big(t;\Gamma(t)\Big) \geq 0$ in every continuity point of $t \mapsto \ell(t)$

- $\dot{\ell}(t)\,\Big(1 - G\Big(t;\Gamma(t)\Big)\Big) = 0$

The case of many noninteracting cracks



$M \geq 2$ fixed, $\quad m = 1, \ldots, M$

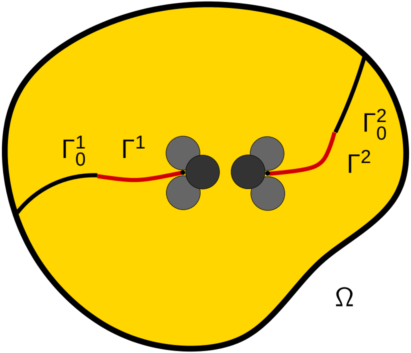
Initial crack: $\Gamma_0 := \Gamma_0^1 \cup \ldots \cup \Gamma_0^M$

Admissible cracks: $\Gamma := \Gamma^1 \cup \ldots \cup \Gamma^M$

$\Gamma^m \supseteq \Gamma_0^m$ for every m, of class $C^{1,1}$

Condition on the two disks of radius $\eta > 0$

The case of many noninteracting cracks



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$+$ Condition of distance between the tips

$\rightarrow$ Class of admissible cracks $\mathcal{R}_\eta^M$

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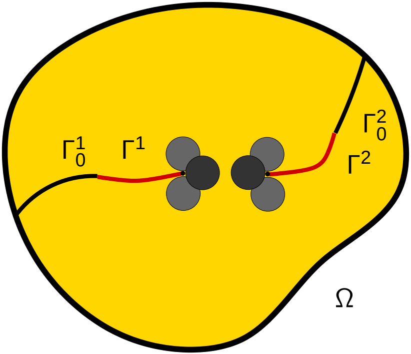

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$\longrightarrow$ Class of admissible cracks $\mathcal{R}_\eta^M$

Griffith's criterion

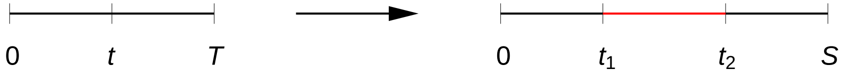
- $\dot\ell^m(t) \geq 0$
- $1 - G\Big(t; \Gamma^m(t)\Big) \geq 0$ in every continuity point of $t \mapsto \ell^m(t)$
- $\dot\ell^m(t)\,\Big(1 - G\Big(t; \Gamma^m(t)\Big)\Big) = 0$

Behaviour at the jump points

Fact: The evolutions $t \mapsto \Gamma^m(t)$ are not continuous and present some jumps

Idea: Reparametrize the time interval, freezing the time at the jump points

Introduce a fictitious time scale that grows with the crack during the jump



t jump point $\mapsto$ interval $[t_1, t_2]$

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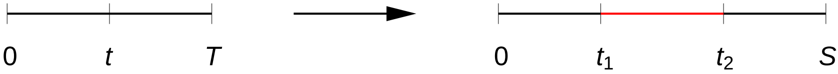

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Reparametrized evolution: $[0, S] \ni s \mapsto (\Gamma^1(s), \ldots, \Gamma^M(s))$

- In the continuity points: Griffith's criterion (rate-independent)

$$G^m \leq 1, \quad G^m = 1 \text{ during the growth}$$

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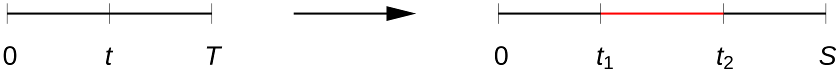

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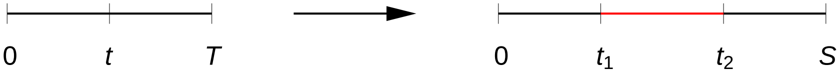

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- In the jump points: $G^m \geq 1$ (for the curves Γ^m that are jumping)

$$\dot{\ell}^m(s) = G\Big(s; \Gamma^m(s)\Big) - 1 \quad \text{during the (reparametrized) jump}$$

The jump is described by a viscous equation (rate-dependent)

Conclusion

- We determine the quasistatic evolution of cracks using Griffith's criterion
- We consider local minimizers employing the method of vanishing viscosity
- We give no a-priori prescriptions on the path followed by the crack
- The behaviour after a jump is known by a viscous equation
- The viscous regime models a fast dynamic transition