A model for crack propagation based on viscous approximation

Giuliano Lazzaroni

Institut d’Alembert, UPMC

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Joint work with Rodica Toader (Udine)
Aim of the work

- **Crack propagation:** We want to determine the evolution of a *brittle* body
  
  **Griffith’s principle:** The crack’s energy is proportional to its measure

- **Viscous approximation:** Standard method for *rate-independent* processes
  
  At every instant the configuration is a *stationary point* for the total energy

- **Differently from previous existence results** based on local minimization,
  
  we give *no a-priori prescriptions on the path* followed by the crack
Crack growth in brittle materials

$\Omega \subseteq \mathbb{R}^2$ section of an elastic body

$\Gamma \subseteq \Omega$ crack of measure $\ell(\Gamma) < +\infty$

$u : \Omega \setminus \Gamma \to \mathbb{R}$ displacement of the elastic part

Antiplane setting: $(x_1, x_2, x_3) \mapsto (x_1, x_2, u(x_1, x_2) + x_3)$
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Boundary condition: $u = \psi(t)$ on $\partial \Omega$
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Problem

Find \( t \mapsto (u(t), \Gamma(t)) \) such that \((u(t), \Gamma(t))\) is an equilibrium for \( E \) taking into account the irreversibility of fracture.
Equilibrium configurations

\[ \Omega \subseteq \mathbb{R}^2 \] section of an elastic body

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\[ u(t; \Gamma) := \arg\min \left\{ \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u(x)|^2 \, dx : \ u = \psi(t) \text{ on } \partial \Omega \right\} \]
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Bulk energy \( W(t; \Gamma) := \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u(t; \Gamma)|^2 \, dx \)

Crack energy \( \ell(\Gamma) := \text{length of } \Gamma \)

Total energy \( E(t; \Gamma) := W(t; \Gamma) + \ell(\Gamma) \)
Equilibrium configurations

Problem

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Equilibrium configurations

Problem

Find \( t \mapsto (u(t), \Gamma(t)) \) such that \((u(t), \Gamma(t))\) is an equilibrium for \( \mathcal{E} \) taking into account the irreversibility of fracture.

Unilateral equilibria (Griffith, 1920)

\( \Gamma(t) \) should be stationary for \( \mathcal{E}(t; \Gamma) \) on the set \( \{ \Gamma: \Gamma \supseteq \Gamma(t) \} \)
Equilibrium configurations

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Difficulty

The notion of critical point is not defined in this case.

There are two main possible approaches:

- Restrict to global minima instead of all stationary points
- Give some prescriptions on the crack path allowing for local notions
The Francfort-Marigo model

Assumption: For every $t \in [0, T]$ the crack $\Gamma(t)$ is a unilateral global minimizer
\[ \mathcal{E}(t; \Gamma(t)) \leq \mathcal{E}(t; \Gamma) \quad \forall \Gamma \supseteq \Gamma(t) \]


Advantages:
- This notion is well defined for a wide class of cracks (closed or rectifiable)
- Crack path determined by the energy criterion, not prescribed a priori
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Existence results

- Dal Maso-Toader (2002): Antiplane elasticity
- Chambolle (2003): Planar elasticity
- Francfort-Larsen (2003): Rectifiable cracks, \( SBV \) deformations
- Dal Maso-Francfort-Toader (2005): Nonlinear elasticity
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Drawback: Jumps between different energy wells (because of nonconvexity)
The Francfort-Marigo model

Global minimizers $\Rightarrow$ Jumps between energy wells, overtaking barriers

$t_1 \geq 0$
Global minimizers $\Rightarrow$ Jumps between energy wells, overtaking barriers
The Francfort-Marigo model

Global minimizers $\Rightarrow$ Jumps between energy wells, overtaking barriers

Alternative: Allow just for overtaking “small” energy barriers (Larsen, 2010)
Local minimization

Local minimization $\Rightarrow$ Jumps happen later, without overtaking barriers

$t_1 \geq 0$

$t_2 > t_1$
Local minimization

Local minimization $\Rightarrow$ Jumps happen later, without overtaking barriers

$$E(\Gamma)(t_1) \geq 0$$

$$E(\Gamma)(t_3) > t_2$$
Local minimization

Problem

Find $t \mapsto (u(t), \Gamma(t))$ such that $(u(t), \Gamma(t))$ is an equilibrium for $\mathcal{E}$ taking into account the irreversibility of fracture.

How to define a notion of local minimizer in the space of curves?
Local minimization

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How to define a notion of local minimizer in the space of curves?

Example: Crack evolving on a fixed curve \( \Gamma \) (e.g., \( \Gamma \) a straight line, a smooth curve, \ldots)
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How to define a notion of local minimizer in the space of curves?

Example: Crack evolving on a fixed curve $\Gamma$ (e.g., $\Gamma$ a straight line, a smooth curve, ...)

$\Gamma_0 \subseteq \Gamma$ initial crack

$\Gamma_\ell$ extension of $\Gamma_0$ along $\Gamma$

$\Rightarrow$ The system depends on only one variable: the length $\ell$ of the curve.
Local minimization

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Case of prescribed crack, via vanishing viscosity

## Aim of the work

### Francfort-Marigo model
- crack path **not** prescribed, very wide class
- consider **only** global minimizers

### Approach with prescribed curve
- crack path **known** a priori, smooth enough
- allow for **local** minimization via vanishing viscosity
Aim of the work

Francfort-Marigo model

- crack path not prescribed, very wide class
- consider only global minimizers

Approach with prescribed curve

- crack path known a priori, smooth enough
- allow for local minimization via vanishing viscosity

Goal

- crack path not prescribed (but quite regular)
- allow for local minimization via vanishing viscosity
Energy release rate

**Aim:** Define a derivative \( \partial_{\Gamma} \mathcal{W}(t; \Gamma) \)
Energy release rate

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Energy release rate

**Aim:** Define a derivative “$\partial_{\Gamma} \mathcal{W}(t; \Gamma)$”

- $\Gamma \subseteq \Omega$ smooth prescribed crack
- $\Gamma_0$ portion of the curve $\Gamma$
- $\Gamma_\ell$ extension of $\Gamma_0$ of length $\ell$
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**Elastic energy**

$$\mathcal{W}(t; \Gamma_\ell) := \frac{1}{2} \int_{\Omega \setminus \Gamma_\ell} |\nabla u_\ell|^2$$

**Energy release rate**

$$G(t; \Gamma_0) := -\partial_\ell \mathcal{W}(t; \Gamma_\ell) |_{\ell = \ell(\Gamma_0)}$$
**Energy release rate**

**Aim:** Define a derivative \( \partial_\Gamma W(t; \Gamma) \)

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**Energy release rate**

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G(t; \Gamma_0) := -\partial_\ell W(t; \Gamma_\ell)\big|_{\ell=\ell(\Gamma_0)}
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---

**Fact:** The energy release rate is actually independent of the continuation of \( \Gamma_0 \)
Griffith’s criterion

Length: $\ell(t) := \ell(\Gamma(t))$
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- \( \dot{\ell}(t) \geq 0 \) (crack growth is irreversible)
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- \( \dot{\ell}(t) \geq 0 \) (crack growth is irreversible)
- \( G(t; \Gamma(t)) \leq 1 \) (the rate can’t exceed the toughness 1)
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Length: $\ell(t) := \ell(\Gamma(t))$

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- $\dot{\ell}(t) \geq 0$ (crack growth is irreversible)
- $G(t; \Gamma(t)) \leq 1$ (the rate can’t exceed the toughness 1)
- $[1 - G(t; \Gamma(t))] \dot{\ell}(t) = 0$ (the crack grows only if the rate equals 1)
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Total energy: \( \mathcal{E}(t; \Gamma) := \mathcal{W}(t; \Gamma) + \ell(\Gamma) \)

Equivalent statement of Griffith’s criterion

- Irreversibility: \( \dot{\ell}(t) \geq 0 \)
- Unilateral stability: \( \ell(t) \) is stationary for \( \mathcal{E}(t; \Gamma) \) among all \( \Gamma: \ell(\Gamma) \geq \ell(t) \)
- Energy balance: \( \frac{d}{dt} \mathcal{E}(t; \Gamma(t)) \) equals the power of the surface forces
The strategy of vanishing viscosity

Initial data: \( \Gamma_0 \) crack at time \( t = 0 \)
The strategy of vanishing viscosity

1. Initial data: $\Gamma_0$ crack at time $t = 0$

2. Time discretization: $(t_{n,0}, t_{n,1}, \ldots, t_{n,n-1}, t_{n,n})$ such that

$$0 = t_{n,0} < \ldots < t_{n,n} = T$$

and

$$\max_{1 \leq i \leq n} (t_{n,i} - t_{n,i-1}) = 0 \quad \text{as} \quad n \to \infty$$
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3. Introduce a small viscous parameter \( \varepsilon \)
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4. Define some approximate solutions:
   - \( \Gamma_{\varepsilon,n,0} := \Gamma_0 \)
   - \( \Gamma_{\varepsilon,n,i} \) a solution to the viscous problem at time \( t_{n,i} \)
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5. Time-continuous limit as $n \to \infty$ with $\varepsilon$ fixed: obtain viscous solutions

For the limit: select a suitable class of admissible cracks
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5. Time-continuous limit as $n \to \infty$ with $\varepsilon$ fixed: obtain viscous solutions

6. Inviscid limit as $\varepsilon \to 0$: obtain a rate-independent evolution

7. Griffith’s criterion in the limit (using some continuity of $G(t, \cdot)$)

For the limit: select a suitable class of admissible cracks
Admissible cracks

Initial crack: $\Gamma_0$ simple curve, not tangent to $\partial \Omega$
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Admissible cracks: \( \Gamma \supseteq \Gamma_0 \) of class \( C^{1,1} \)
(without knowing a priori the crack path)
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Condition: The two tangent disks of radius $\eta > 0$
do not intersect the whole curve $\Gamma$ and $\partial \Omega$
This defines the class of admissible cracks $\mathcal{R}_\eta$
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- The curvature is controlled by $\frac{1}{\eta}$
- It is possible to find limits of class $C^{1,1}$ in $\mathcal{R}_\eta$
- In the limit, the tip does not meet the curve $\Gamma$, nor the boundary of $\Omega$
Admissible cracks

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- It is possible to find limits of class $C^{1,1}$ in $\mathcal{R}_\eta$
- In the limit, the tip does not meet the curve $\Gamma$, nor the boundary of $\Omega$
- We have convergence of parametrizations, tangents, and lengths
- We have convergence of the energies and of the energy release rates
Approximation

Time discretization: \((t_{n,0}, t_{n,1}, \ldots, t_{n,n-1}, t_{n,n})\) such that

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Approximate solutions

- \(\Gamma_{\varepsilon,n,0} := \Gamma_0\)
- \(\Gamma_{\varepsilon,n,i}\) a solution to

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\min_{\Gamma_{\varepsilon,n,i-1} \subseteq \Gamma \in \mathcal{R}_\eta} \left\{ \mathcal{W}(t_{n,i}; \Gamma) + \ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1}) + \frac{\varepsilon}{2} \frac{\ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}} \right\}
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1. Time-continuous limit: \(\varepsilon > 0, \ n \to \infty\)

Then \(\Gamma_{\varepsilon,n,i}\) converges to a continuous function \(t \mapsto \Gamma_{\varepsilon}(t)\)
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   Then \(\Gamma_{\varepsilon,n,i}\) converges to a continuous function \(t \mapsto \Gamma_\varepsilon(t)\)

2. **Rate-independent limit:** \(\varepsilon \to 0\)
   Then \(\Gamma_\varepsilon(t)\) converges to a function \(t \mapsto \Gamma(t)\)
Approximation

Time discretization: \((t_{n,0}, t_{n,1}, \ldots, t_{n,n-1}, t_{n,n})\) such that
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1. **Time-continuous limit:** \(\varepsilon > 0, \ n \to \infty\)
   Then \(\Gamma_{\varepsilon,n,i}\) converges to a continuous function \(t \mapsto \Gamma_{\varepsilon}(t)\)

2. **Rate-independent limit:** \(\varepsilon \to 0\)
   Then \(\Gamma_{\varepsilon}(t)\) converges to a function \(t \mapsto \Gamma(t)\) with jumps
Passage to the limit

Start with the initial condition
Passage to the limit

Solve inductively the \textit{viscous} problems in $\mathcal{R}_\eta$

$\Gamma_{\varepsilon,n,i} \in \mathcal{R}_\eta$
Passage to the limit

Solve inductively the viscous problems in $\mathcal{R}_\eta$

$\Gamma_{\epsilon,n,i} \in \mathcal{R}_\eta$

Consider the piecewise constant interpolations $\Gamma_{\epsilon,n}(t)$
Passage to the limit

Obtain an approximate sequence

$$\Gamma_{\varepsilon,n}(t) \in \mathcal{R}_\eta$$
Passage to the limit

Time-continuous limit at $\varepsilon$ fixed

$$\Gamma_{\varepsilon,n}(t) \rightarrow \Gamma_{\varepsilon}(t) \quad \text{in} \ \mathcal{R}_\eta$$

$$\mathcal{E}(t; \Gamma_{\varepsilon,n}(t)) \rightarrow \mathcal{E}(t; \Gamma_{\varepsilon}(t)), \quad G(t; \Gamma_{\varepsilon,n}(t)) \rightarrow G(t; \Gamma_{\varepsilon}(t))$$
Passage to the limit

Obtain a sequence of *viscous solutions*

\[ \Gamma_\varepsilon(t) \in \mathcal{R}_\eta \]
Passage to the limit

Rate-independent limit

\[ \Gamma_\varepsilon(t) \to \Gamma(t) \quad \text{in} \ R_\eta \]

\[ \mathcal{E}(t; \Gamma_\varepsilon(t)) \to \mathcal{E}(t; \Gamma(t)) , \quad G(t; \Gamma_\varepsilon(t)) \to G(t; \Gamma(t)) \]
Griffith’s criterion

- $\Gamma_{\varepsilon,n,i}$ minimizes $W(t_n,i; \Gamma) + \ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1}) + \frac{\varepsilon}{2} \frac{\ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1})^2}{t_n,i - t_{n,i-1}}$
Griffith’s criterion

- $\Gamma_{\varepsilon,n,i}$ minimizes $\mathcal{W}(t_{n,i}; \Gamma) + \ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1}) + \frac{\varepsilon}{2} \frac{\ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}}$

- Passing to the derivatives

$$-G\left(t; \Gamma_{\varepsilon,n}(t)\right) + 1 + \varepsilon \dot{\ell}_{\varepsilon,n}(t) \geq 0$$
Griffith’s criterion

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- Passing to the derivatives

$$-G\left(t; \Gamma_{\varepsilon, n}(t)\right) + 1 + \varepsilon \dot{\ell}_{\varepsilon, n}(t) \geq 0$$

- We have the equality when $\dot{\ell}_{\varepsilon, n}(t) > 0$

Discrete Griffith’s criterion

- $\dot{\ell}_{\varepsilon, n}(t) \geq 0$

- $1 - G\left(t; \Gamma_{\varepsilon, n}(t)\right) + \varepsilon \dot{\ell}_{\varepsilon, n}(t) \geq 0$

- $\dot{\ell}_{\varepsilon, n}(t) \left(1 - G\left(t; \Gamma_{\varepsilon, n}(t)\right) + \varepsilon \dot{\ell}_{\varepsilon, n}(t)\right) = 0$
Griffith’s criterion

• $\Gamma_{\varepsilon,n,i}$ minimizes $\mathcal{W}(t_{n,i}; \Gamma) + \ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1}) + \frac{\varepsilon}{2} \frac{\ell(\Gamma \setminus \Gamma_{\varepsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}}$

• Passing to the derivatives

$$-G(t; \Gamma_{\varepsilon,n}(t)) + 1 + \varepsilon \dot{\ell}_{\varepsilon,n}(t) \geq 0$$

• We have the equality when $\dot{\ell}_{\varepsilon,n}(t) > 0$

• Then we pass to the limit

Viscous Griffith’s criterion

• $\dot{\ell}_{\varepsilon}(t) \geq 0$

• $1 - G(t; \Gamma_{\varepsilon}(t)) + \varepsilon \dot{\ell}_{\varepsilon}(t) \geq 0$

• $\dot{\ell}_{\varepsilon}(t) \left(1 - G(t; \Gamma_{\varepsilon}(t)) + \varepsilon \dot{\ell}_{\varepsilon}(t)\right) = 0$
Griffith’s criterion

- $\Gamma_{\epsilon,n,i}$ minimizes

$$W(t_{n,i}; \Gamma) + \ell(\Gamma \setminus \Gamma_{\epsilon,n,i-1}) + \frac{\epsilon}{2} \frac{\ell(\Gamma \setminus \Gamma_{\epsilon,n,i-1})^2}{t_{n,i} - t_{n,i-1}}$$

- Passing to the derivatives

$$-G(t; \Gamma_{\epsilon,n}(t)) + 1 + \epsilon \dot{\ell}_{\epsilon,n}(t) \geq 0$$

- We have the equality when $\dot{\ell}_{\epsilon,n}(t) > 0$

- Then we pass to the limit

Griffith’s criterion

- $\dot{\ell}(t) \geq 0$

- $1 - G(t; \Gamma(t)) \geq 0$ in every continuity point of $t \mapsto \ell(t)$

- $\dot{\ell}(t) \left(1 - G(t; \Gamma(t))\right) = 0$
The case of many noninteracting cracks

\[ M \geq 2 \text{ fixed, } \quad m = 1, \ldots, M \]

Initial crack: \[ \Gamma_0 := \Gamma_0^1 \cup \ldots \cup \Gamma_0^M \]

Admissible cracks: \[ \Gamma := \Gamma^1 \cup \ldots \cup \Gamma^M \]
\[ \Gamma^m \supseteq \Gamma_0^m \text{ for every } m, \quad \text{of class } C^{1,1} \]

Condition on the two disks of radius \( \eta > 0 \)
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Condition on the two disks of radius \( \eta > 0 \)

\( + \) Condition of distance between the tips

\( \rightarrow \) Class of admissible cracks \( \mathcal{R}_\eta^M \)
The case of many noninteracting cracks

$\Omega$

$\Gamma_0 \cup \ldots \cup \Gamma_0$

$\Gamma_1 \cup \ldots \cup \Gamma_1$

$\Gamma_2 \cup \ldots \cup \Gamma_2$

$M \geq 2$ fixed, $m = 1, \ldots, M$

Initial crack: $\Gamma_0 := \Gamma_0^1 \cup \ldots \cup \Gamma_0^M$

Admissible cracks: $\Gamma := \Gamma^1 \cup \ldots \cup \Gamma^M$

$\Gamma^m \supseteq \Gamma^m_0$ for every $m$, of class $C^{1,1}$

Condition on the two disks of radius $\eta > 0$

+ Condition of distance between the tips

$\rightarrow$ Class of admissible cracks $R^M_\eta$

Griffith’s criterion

- $\dot{\ell}^m(t) \geq 0$
- $1 - G(t; \Gamma^m(t)) \geq 0$
- $\dot{\ell}^m(t) \left(1 - G(t; \Gamma^m(t))\right) = 0$

in every continuity point of $t \mapsto \ell^m(t)$
Behaviour at the jump points

**Fact:** The evolutions $t \mapsto \Gamma^m(t)$ are **not** continuous and present some jumps

**Idea:** Reparametrize the time interval, freezing the time at the jump points

Introduce a fictitious time scale that grows with the crack during the jump

![Diagram](image)

$t$ jump point $\mapsto$ interval $[t_1, t_2]$
Behaviour at the jump points

Fact: The evolutions $t \mapsto \Gamma^m(t)$ are not continuous and present some jumps

Idea: Reparametrize the time interval, freezing the time at the jump points
Introduce a fictitious time scale that grows with the crack during the jump

Reparametrized evolution: $[0, S] \ni s \mapsto (\Gamma^1(s), \ldots, \Gamma^M(s))$

- In the continuity points: Griffith’s criterion (rate-independent)

\[ G^m \leq 1, \quad G^m = 1 \text{ during the growth} \]
Behaviour at the jump points

Fact: The evolutions $t \mapsto \Gamma^m(t)$ are not continuous and present some jumps

Idea: Reparametrize the time interval, freezing the time at the jump points
Introduce a fictitious time scale that grows with the crack during the jump

\[\begin{array}{c}
0 & t & T \\
\end{array}\quad \begin{array}{c}
0 & t_1 & t_2 & S \\
\end{array}\]

$t$ jump point $\mapsto$ interval $[t_1, t_2]$

Reparametrized evolution: $[0, S] \ni s \mapsto (\Gamma^1(s), \ldots, \Gamma^M(s))$

- In the continuity points: Griffith’s criterion (rate-independent)
  \[G^m \leq 1, \quad G^m = 1 \text{ during the growth}\]
- In the jump points: $G^m \geq 1$ (for the curves $\Gamma^m$ that are jumping)
Behaviour at the jump points

**Fact:** The evolutions \( t \mapsto \Gamma^m(t) \) are **not** continuous and present some **jumps**

**Idea:** Reparametrize the time interval, freezing the time at the jump points
Introduce a fictitious time scale that grows with the crack during the jump

Reparametrized evolution: \([0, S] \ni s \mapsto (\Gamma^1(s), \ldots, \Gamma^M(s))\)

- **In the continuity points:** Griffith’s criterion (**rate-independent**)
  \[ G^m \leq 1, \quad G^m = 1 \text{ during the growth} \]

- **In the jump points:** \( G^m \geq 1 \) (for the curves \( \Gamma^m \) that are jumping)
  \[ \dot{\ell}^m(s) = G(s; \Gamma^m(s)) - 1 \quad \text{during the (reparametrized) jump} \]

The jump is described by a **viscous equation** (**rate-dependent**)
Conclusion

- We determine the *quasistatic* evolution of cracks using Griffith’s criterion.
- We consider **local minimizers** employing the method of vanishing viscosity.
- We give no a-priori prescriptions on the path followed by the crack.
- The behaviour after a **jump** is known by a **viscous equation**.
- The viscous regime models a **fast dynamic** transition.