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An optimization approach for the Cauchy problem in linear elasticity

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Abstract An energy error functional is introduced in the context of the ill-posed problem of boundary data recovery in linear elasticity, which is well known as the Cauchy problem. The problem is converted into one of optimization; the computation of the gradients of the energy functional is given for both the continuous and the discrete problems. Links with existing methods for data completion are described and numerical experiments highlight the efficiency of the proposed method as well as its robustness in the case of singular data.

Keywords *Cauchy problem, Data completion, Optimization, Identification, Adjoint derivative, Inverse problem.*

1 Introduction

Nowadays, the non-destructive inspection of material in order to detect internal damage, inclusions, material properties, surface traction and displacement intensity is becoming increasingly important in industry. These problems involve the solution of inverse problems by using overspecified data measured on outer or easily accessible boundaries. As an alternative to usual least-squares error formulations, we present a new approach using only the available data on a part of the boundary, which allows the reconstruction of the missing data relating to the internal or inaccessible boundaries or even on arbitrary internal surfaces inside a solid. This type of inverse problem is sometimes called data completion.

Since Hadamard (1953), the data completion problem, often also known as the Cauchy problem, has been considered ill-posed in the sense that the dependence of the lacking data on the measured data is not continuous. There is a wealth of literature dealing with Cauchy problems, in which various algorithms have been proposed. The solution of general inverse problems was discussed in Tikhonov and Arsenin (1977). A survey of inverse problems in structural mechanics was proposed by Bui (1994) while Kozlov *et al* (1991) proposed an iterative algorithm to solve data completion for the Laplace equation and Lamé system. Lesnic *et al.* (2004) used it to solve Cauchy problems in the context of elasticity and thermostatics.

This paper presents a new computational algorithm for the reconstruction of lacking data based on the minimization of an energy functional. This work extends an approach already used for the Laplace problem to that of linear elasticity; see Andrieux *et al* (2006). The method proposed has already been introduced in the framework of parameter identification; see for instance the works of Kohn and Vogelius (1985, 1987) and those of Ladevèze *et al* (1993, 1994). By using the overspecified data measured on an accessible part of the boundary, the method proposed minimizes a functional whose design variables are the missing data (Neumann and Dirichlet). This functional is defined as an energy error between the solutions of two well-posed problems: the first has measured Dirichlet boundary data on the accessible part of the boundary and the unknown Neumann boundary data on the other part; the second has measured Neumann boundary data on the accessible part of the boundary and unknown Dirichlet boundary data on the other part.

The computation of the gradients of the energy functional is given for both the continuous and the discrete problems. The link with the method proposed by Kozlov *et al* (1991) is established. To explore the efficiency of this method and its robustness in the case of singular data, numerical example is presented for 2D. The results are in good agreement with those obtained by the exact solution. The method is both very efficient with respect to the accuracy of the solution and regarding the amount of computation needed. The formulation is very general and can be used with heterogeneous materials and other physical phenomena.

2 Statement of the problem

We consider a linear elastic material which occupies an open bounded domain Ω with a smooth boundary Γ as shown on figure 1. Γ_m and Γ_u, respectively denote the boundary where the overspecified data can be measured and the boundary where the data have to be recovered. The stiffness tensor of the material, which can be a field depending on the space variable, is denoted $\mathbb{C}$.

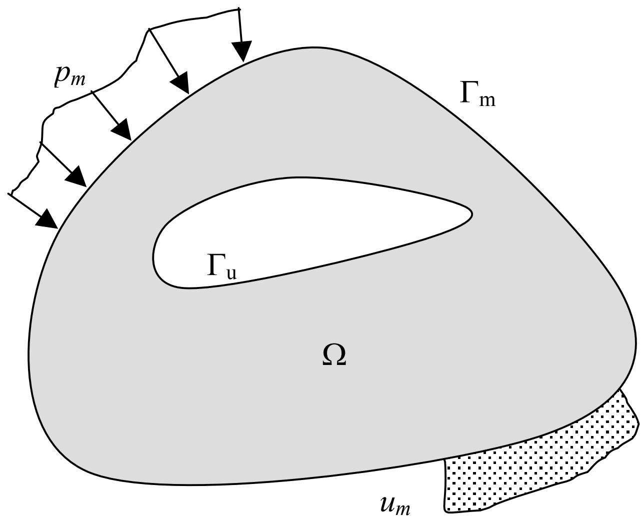

Fig.1: Elastic body Ω with boundary $\Gamma = \Gamma_m \cup \Gamma_u$. p_m is the measured pressure and u_m the measured displacements field.

Given the displacements and the surface tractions on Γ_m, it is necessary to recover the corresponding displacement and surface tractions on the remaining part of boundary Γ_u. Knowing that an equilibrium elastic displacement field exists with these data the problem can be stated as follows:

Find *(f, d)* on Γ_u such that a displacement field u exists satisfying:

$$\begin{cases} div(\sigma(u)) = 0 \quad in\ \Omega \\ \sigma(u) = \mathbb{C} : \varepsilon(u)\ in\ \Omega \\ u = u_m \qquad\qquad on\, \Gamma_m \\ \sigma(u).n = p_m \quad on\, \Gamma_m \end{cases} \tag{1}$$

where p_m is the measured surface traction vector, u_m is the measured displacement field, σ is the stress tensor and ε is the strain tensor in Ω.

3 Data completion

Let us consider the Cauchy problem (1). Assuming that the data (p_m, u_m) are compatible, i.e. that this pair is indeed the trace and the surface traction associated with a unique elastic equilibrium field *u,* the data is extended such that:

$$\begin{cases} div(\sigma(u)) = 0 \qquad\qquad in\ \Omega \\ \sigma(u) = \mathbb{C} : \varepsilon(u) \qquad\quad in\ \Omega \\ u = u_m\, ,\ \sigma(u).n = p_m \quad on\ \Gamma_m \\ u = d\, ,\ \sigma(u).n = f \qquad on\, \Gamma_u \end{cases} \tag{2}$$

This problem referred to as the Cauchy problem, is much more difficult to solve both analytically and numerically than the direct problem, since the solution does not satisfy the general conditions of well-posedness. Various methods have been proposed to solve this ill-posed problem. Almost all of them are iterative methods that repeat the solutions of direct, classical well-posed problems. Classical least square methods are commonly used, as are more elaborate ones as in Koslov *et al.* (1991), or Cimetiere *et al.* (2001). It should be noticed that the amount of numerical computation needed is a decisive issue if realistic and accurate 3D applications are considered.

3.1 Energy error functional

In this paper, we propose the direct recovery of the lacking data (f, d) using an energy error functional previously introduced in the framework of parameter identification; see for instance the review paper of Bonnet and Constantinescu (2005). Our approach is based on the definition of two displacement fields u_1 and u_2, the solutions of two classical elastic problems (3) and (4) respectively, and which are related to a given surface traction field η and a given displacement surface field τ on boundary Γ_u of the solid. This is done by satisfying only one of the known boundary data on the remaining part of the boundary Γ_m:

$$\begin{cases} div(\sigma(u_1)) = 0 \quad in\ \Omega \\ \sigma(u_1) = \mathbb{C} : \varepsilon(u_1)\ \ in\ \Omega \\ u_1 = u_m \qquad\qquad on\ \Gamma_m \\ \sigma(u_1).n = \eta \qquad on\, \Gamma_u \end{cases} \tag{3}$$

$$\begin{cases} div(\sigma(u_2)) = 0 & in\ \Omega \\ \sigma(u_2) = \mathbb{C} : \varepsilon(u_2) & in\ \Omega \\ u_2 = \tau & on\ \Gamma_u \\ \sigma(u_2).n = p_m & on\,\Gamma_m \end{cases} \tag{4}$$

By observing that if these two fields coincide: $u_1 = u_2 = u$, then $(\eta,\ \tau)$ are the solutions of the original data completion problem (2). We propose to build an energy error functional by measuring the gap between the fields $(u_1 - u_2)$:

$$E(u_1, u_2, \eta, \tau) = \frac{1}{2}\int_\Omega \sigma(u_1 - u_2) : \varepsilon(u_1 - u_2) = \frac{1}{2}\int_\Omega \mathbb{C} : \varepsilon(u_1 - u_2) : \varepsilon(u_1 - u_2)$$

$$with\ \ u_1 = u_1(u_m, \eta) \qquad u_2 = u_2(p_m, \tau) \tag{5}$$

Then we propose to solve the data completion problem by minimizing the function *E*. The following minimization problem can be stated:

$$(f, d) = \underset{(\eta,\tau)}{\arg\min}\, E(u_1(\ u_m, \eta), u_2(p_m, \tau))$$

$$with \quad u_1\,(u_m, \eta)\ solution\ of\ (3) \qquad u_2\,(p_m, \tau)\ solution\ of\ (4) \tag{6}$$

The justification of this formulation lies in the following properties:

- ✓ *E* is a positive functional,
- ✓ if a pair of fields (η, τ) satisfies $E(\eta, \tau) = 0$ then:
  - a. $u_2 = u_1 + R$, where *R* is a solid body displacement field,
  - b. u_1 and (η, τ) solve the equations (2).
- ✓ *E* is a convex quadratic functional with respect to (η, τ).

By using the properties of u_1 and u_2, it is easy to derive an alternative expression of *E*:

$$E(u_1, u_2, \eta, \tau) = \frac{1}{2}\int_{\partial\Omega} \sigma(u_1 - u_2).n.(u_1 - u_2) \tag{7}$$

Thus:

$$E(u_1, u_2, \eta, \tau) = \frac{1}{2}\int_{\Gamma_m} (\sigma(u_1).n - p_m).(u_m - u_2) + \frac{1}{2}\int_{\Gamma_u} (\eta - \sigma(u_2).n).(u_1 - \tau) \tag{8}$$

The above expression shows that the energy error can be expressed only with a boundary error term, which is the term we use for its computation. Unlike in the least-squares methods, the error on the surface tractions is weighted by the error on the displacements. This feature illustrates the more symmetric formulation that can be obtained by an error functional based on the energy of the problem: no dimensional coefficient is needed, whereas this would be an essential feature if a mixed least-squares functional were used.

3.2 The energy error derivative

The resolution of the data completion problem is based on the minimization of the functional E, and efficient minimization is achieved with a trust region descent algorithm that uses the gradient of the functional. Thanks to the superposition principle, fields u_1 and u_2 are affine with respect to the data pair (η,τ), thereby facilitating the direct evaluation of the directional derivatives of E. For every pair of (ψ,h), we obtain:

$$\begin{cases} \dfrac{\partial E}{\partial \eta}\cdot\psi = \displaystyle\int_{\Gamma_u} \left(u_1(\eta)-\tau\right)\cdot\psi + \int_{\Gamma_m} \left(u_m-u_2(\tau)\right)\sigma(u_1^*(\psi))\cdot n \\ \dfrac{\partial E}{\partial \tau}\cdot h = \displaystyle\int_{\Gamma_u} \left(\sigma(u_2(\tau))\cdot n-\eta\right)\cdot h + \int_{\Gamma_m} \left(p_m-\sigma(u_1(\eta))\cdot n\right)u_2^*(h) \end{cases} \quad (9)$$

where u_1^* and u_2^* solve (10) and (11):

$$\begin{cases} div(\sigma(u_1^*))=0 & in\ \Omega \\ \sigma(u_1^*)=\mathbb{C}:\varepsilon(u_1^*) & in\ \Omega \\ u_1^*=0 & on\ \Gamma_m \\ \sigma(u_1^*).n=\psi & on\,\Gamma_u \end{cases} \quad (10)$$

$$\begin{cases} div(\sigma(u_2^*))=0 & in\ \Omega \\ \sigma(u_2^*)=\mathbb{C}:\varepsilon(u_2^*) & in\ \Omega \\ u_2^*=h & on\ \Gamma_u \\ \sigma(u_2^*).n=0 & on\,\Gamma_m \end{cases} \quad (11)$$

3.3 The adjoint derivative

The components of the gradient can nevertheless be computed more efficiently by using the classical adjoint method. The computation requires the determination of two adjoint displacement fields v_1 and v_2, which are the solutions of well-posed problems described previously and very similar to the direct problem used for the definition of the u_1 and u_2 fields. We consider the following Lagrangian equation:

$$\begin{aligned} L(u_1,u_2,v_1,v_2,\lambda,\mu;\eta,\tau) &= E(u_1,u_2,\eta,\tau)+\int_\Omega \sigma(u_1):\varepsilon(v_1)-\int_{\Gamma_u}\eta v_1+\cdots \\ & \int_\Omega \sigma(u_2):\varepsilon(v_2)-\int_{\Gamma_m} p_m v_2+\int_{\Gamma_u}\left(\mu v_2+\lambda(u_2-\tau)\right) \end{aligned} \quad (12)$$

With: $(u_1,u_2,v_1,v_2,\lambda,\mu)\in V_1\times H^1(\Omega)\times V_1^0\times H^1(\Omega)\times H^{-1/2}(\Gamma_u)\times H^{-1/2}(\Gamma_u)$
where:

$$\begin{aligned} V_1 &= \left\{v\in H^1(\Omega)/v\,|_{\Gamma_m}=u_m\right\} \\ V_1^0 &= \left\{v\in H^1(\Omega)/v\,|_{\Gamma_m}=0\right\} \end{aligned} \quad (13)$$

For any (η,τ) and the above fields we obtain:

$$E(\eta,\tau)=L(u_1,u_2,v_1,v_2,\lambda,\mu;\eta,\tau) \quad (14)$$

Hence, the gradient of E can be obtained from the partial derivatives of L with respect to η and τ:

$$\frac{\partial E(\eta,\tau)}{\partial \eta}\cdot\psi = -\int_{\Gamma_u} v_1\cdot\psi \tag{15}$$

$$\frac{\partial E(\eta,\tau)}{\partial \tau}\cdot h = -\int_{\Gamma_u} \left(\eta - \sigma(u_2)\cdot n - \sigma(v_2)\cdot n\right)h \tag{16}$$

In these equations, the interpretation of the Lagrange multiplier λ obtained from the stationary condition of the Lagrangian equation with respect to the primal field u_2 has been taken into account in order to derive expressions involving the v_2 and u_2 fields. The adjoint fields involved are solutions of the following boundary value problems:

$$\begin{cases} div(\sigma(v_1)) = 0 & in\ \Omega \\ \sigma(v_1) = \mathbb{C} : \varepsilon(v_1) & in\ \Omega \\ v_1 = 0 & on\ \Gamma_m \\ \sigma(v_1).n = \sigma(u_2).n - \eta & on\,\Gamma_u \end{cases} \tag{17}$$

$$\begin{cases} div(\sigma(v_2)) = 0 & in\ \Omega \\ \sigma(v_2) = \mathbb{C} : \varepsilon(v_2) & in\ \Omega \\ v_2 = 0 & on\ \Gamma_u \\ \sigma(v_2).n = \sigma(u_1).n - p_m & on\,\Gamma_m \end{cases} \tag{18}$$

In this Lagrangian form the boundary conditions are incorporated explicitly in order to ensure that the underlying function spaces are fixed and do not depend on the unknown field τ.

3.4 The alternating direction minimization method

The method proposed in this work is related to that introduced by Kozlov *et al* (1991) and subjected to extensive numerical experiments by Lesnic *et al* (2004) and references therein. In this approach, the data completion problem is solved on the basis of an alternating iterative procedure, where successive solutions of well-posed mixed boundary value problems for the original equation are computed. The method has been proven to be convergent. Our approach generalizes that of Kozlov *et al*: as shown below, it turns out to be an alternating minimization procedure in the η and τ directions for the energy error functional introduced here.

In fact problem (2) can be split into two well-posed sub-problems with mixed boundary conditions. For one of them, (Dirichlet, Neumann) conditions are imposed on (Γ_m, Γ_u):

$$\begin{cases} div(\sigma(u_1^{2j})) = 0 & in\ \Omega \\ \sigma(u_1^{2j}) = \mathbb{C} : \varepsilon(u_1^{2j}) & in\ \Omega \\ u_1^{2j} = u_m & on\ \Gamma_m \\ \sigma(u_1^{2j}).n = \eta^{2j} & on\,\Gamma_u \end{cases} \tag{19}$$

while this is reversed for the other and (Neumann, Dirichlet) conditions are imposed on (Γ_m, Γ_u):

$$
\begin{cases} div(\sigma(u_2^{2j+1})) = 0 & in\ \Omega \\ \sigma(u_2^{2j+1}) = \mathbb{C} : \varepsilon(u_2^{2j+1}) & in\ \Omega \\ u_2^{2j+1} = \tau^{2j+1} & on\ \Gamma_u \\ \sigma(u_2^{2j+1}).n = p_m & on\,\Gamma_m \end{cases} \tag{20}
$$

Therefore, solving the Cauchy system (2) is achieved when data completion (*d, f*) leads to the same field $u_1 = u_2$ in Ω. Basically, the iterative data completion procedure of Kozlov *et al* is derived from this observation. It can be summarized as follows: starting from an initial assumption of the pressure η^0 on Γ_u, this assumption is iteratively corrected by alternately solving problems of form (19) and (20), where at each iteration the appropriate boundary data result from the solution of the boundary value problems solved previously. A sequence of well-posed mixed problems is generated as follows: u^{2j+1} solves (20) with τ^{2j+1} substituted by u^{2j}, while u^{2j+2} solves (19) with η^{2j+2} substituted by $\sigma(u^{2j+1}).n$.

By linking the Kozlov *et al* algorithm to the Dirichlet-to-Neumann operator, this procedure is shown to be convergent. The analysis of the convergence of this algorithm for both parabolic and hyperbolic data completion problems by fixed-point arguments can be found in the work of Leitao and Baumeister (2002). Reverting to our energy error functional, Kozlov *et al* algorithm can be interpreted as an alternating-direction minimization method for the energy error functional *E*. More precisely:

- In step *2j+1* of the algorithm: u^{2j+1} is characterized by $u^{2j+1} = u_2(\tau^{2j+1}) \Leftrightarrow \tau^{2j+1} = \arg\min E(\eta^{2j},\tau)\ \ with\ \eta^{2j} = \sigma(u^{2j})\cdot n\,|_{\Gamma_u}$
- In step *2j+2* of the algorithm: u^{2j+2} leads to $u^{2j+2} = u_1(\eta^{2j+2}) \Leftrightarrow \eta^{2j+2} = \arg\min E(\eta,\tau^{2j+1})\ \ with\ \tau^{2j+1} = u^{2j+1}\,|_{\Gamma_u}$

This can be clarified by careful examination of the different steps of the alternating-direction algorithm for the minimization method of $E(\eta, \tau)$, namely minimization with respect to one field, the other being kept fixed.

Minimizing in the τ direction leads to τ_o such that: $\min E(\eta_o,\tau),\ \tau \in H^{1/2}(\Gamma_u)$, where $\eta_o = \sigma(u^{2j}).n$. By applying the first order optimality condition, and using derivative (9), we obtain

$$
\frac{\partial E(\eta_o,\tau_o)}{\partial \tau}\cdot h = 0 \quad \forall\, h \in H^{1/2}(\Gamma_u)
$$

$$
\frac{\partial E(\eta_o,\tau_o)}{\partial \tau}\cdot h = \int_{\Gamma_u} \left[u_2(\tau_o) - u_1(\eta_o)\right]\sigma\left(u_2^*(h)\right).n + \int_{\Gamma_m} \left[u_2(\tau_o) - u_1(\eta_o)\right]\sigma\left(u_2^*(h)\right).n \quad \forall\, h \in H^{1/2}(\Gamma_u)
$$

Recalling that $u_2^*(h)$ satisfies (11), we obtain:

$$
\int_{\Gamma_u} \left[u_1(\eta_o) - u_2(\tau_o)\right]\sigma\left(u_2^*(h)\right)\cdot n \quad \forall\ h \in H^{1/2}(\Gamma_u)
$$

Here we introduce the Poincaré-Stecklov operator:

$$S_1 : H^{1/2}(\Gamma_u) \to H^{-1/2}(\Gamma_u)$$
$$h \quad \to \sigma(u_2^*) \cdot n \tag{21}$$

As $u_2^*(h)$ is the solution of (11) it is easy to check that S_1 is an isomorphism. By using this argument, we obtain the following from (21): $u_2(\tau_o) = u_1(\eta_o)$ on Γ_u

The τ_o, which minimizes $E(\eta_o, \tau)$ is associated with $u_2(\tau_o)$ which turns out to be the u^{2j+1} field solution of (20).

Minimization on η leads to η_o such that: $\min E(\eta, \tau_o)$, $\eta \in H^{-1/2}(\Gamma_u)$, where $\tau_o = u^{2j+1}$. By using the first order optimality condition and derivative (9), we obtain:

$$\frac{\partial E(\eta_o, \tau_o)}{\partial \eta} \cdot \psi = 0 \quad \forall \psi \in H^{-1/2}(\Gamma_u)$$
$$\frac{\partial E(\eta_o, \tau_o)}{\partial \eta} \cdot \psi = \int_{\Gamma_u} \left[\sigma\left(u_2(\tau_o)\right) - \sigma\left(u_1(\eta_o)\right) \right] .n\, u_1^*(\psi)$$
$$+ \int_{\Gamma_m} \left[\sigma\left(u_2(\tau_o)\right) - \sigma\left(u_1(\eta_o)\right) \right] .n\, u_1^*(\psi) \qquad \forall \psi \in H^{-1/2}(\Gamma_u)$$

Recalling that $u_2^*(h)$ satisfies (11), we obtain

$$\int_{\Gamma_u} \left[\sigma\left(u_2(\tau_o)\right) - \sigma\left(u_1(\eta_o)\right) \right] \cdot n\, u_1^*(\psi) \ \ \forall \ \psi \in H^{-1/2}(\Gamma_u)$$

Consideration of the inverse of the Poincaré-Stecklov operator:

$$S_2^{-1} : H^{-1/2}(\Gamma_u) \to H^{1/2}(\Gamma_u)$$
$$\psi \quad \to \ u_1^* \tag{22}$$

as $u_1^*(\psi)$ is the solution of (11), we deduce that:

$$\sigma(u_1(\eta_o)) \cdot n = \sigma(u_2(\tau_o)) \cdot n \ on \ \Gamma_u$$

The convergence of the Kozlov *et al* algorithm is proved for compatible data (p_m, u_m). It should also be noted that better numerical performances can be expected with the present method, since a more efficient optimization algorithm than the alternating directions can be used to perform the minimization of error functional *E*.

4 Discrete approach

The above method was implemented by using the finite element method (FEM). Hence the derivative of the adjoint state (15) is preferably established on the basis of the FEM-discretized problem. The advantage of this fully discrete approach is that the exact gradient of the discrete objective function is obtained; moreover, it is easy to implement.

4.1 Discrete energy error functional

We consider a mesh of Ω characterized by *n* nodes. The number of nodes on boundary Γ_u is denoted by *p* and the number of nodes on boundary Γ_m is denoted by q. By using Lagrange multipliers to formulate the Dirichlet boundary conditions, the discretized forms of the problems (4) are as follows:

$$\begin{bmatrix} K & L_m^T \\ L_m & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ p_1 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ U_m \end{Bmatrix} \quad and \quad \begin{bmatrix} K & L_u^T \\ L_u & 0 \end{bmatrix} \begin{Bmatrix} U_2 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ X_\tau \end{Bmatrix} \tag{23}$$

where K is the overall $n \times n$ stiffness matrix, U_1 and U_2 are the nodal variable vectors and F_1 and F_2 are the load vectors. $\{p_1\}$ and $\{p_2\}$ are the unknown nodal forces related to the degrees of freedom on Γ_m and Γ_u, respectively. The entries of the rectangular indicator $q \times n$ matrix L_m and $p \times n$ matrix L_u are 1 and 0 such that $L_m U_1 = U_m$ and $L_u U_2 = X_\tau$. U_m denotes the discretized form of u_m.
$F_1 = \{X_\eta, 0\}^T$ is the load vector, where X_η contains the nodal unknown loads on Γ_u and depends explicitly on η. The components of U_1 defined on Γ_m are fixed and depend on u_m, the others depend implicitly on η and u_m. Likewise, $U_2 = \{X_\tau, \hat{U}_2\}^T$, where X_τ groups the prescribed nodal variables associated with τ, while $\hat{U}_2$ collects all the nodal variables left unknown by the boundary conditions of (4). The components of load vector F_2 defined on Γ_m are prescribed and depend on p_m, the others are null. The components of load vector p_1 are unknown and depend on X_η and u_m although those of p_2, which are unknown, depend on X_τ and p_m.
However, it should be noted that matrix K is fixed. Thus X_η and X_τ are the discretized design variables of the optimization problem described above (6). The energy function, in its discretized form, can be written as follows:

$$\begin{aligned} E(X_\eta, X_\tau) &= \frac{1}{2}(U_1 - U_2)^T \cdot K(U_1 - U_2) \\ &= \frac{1}{2}(U_1 - U_2)^T \cdot \left(F_1 - L_m^T p_1 - F_2 + L_u^T p_2\right) \end{aligned} \tag{24}$$

The discrete optimization problem is then stated as follows:

$$\begin{aligned} &\min_{(X_\eta, X_\tau)} E\left(X_\eta, X_\tau\right) \\ &with: \\ &\begin{bmatrix} K & L_m^T \\ L_m & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ p_1 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ U_m \end{Bmatrix} \quad and \quad \begin{bmatrix} K & L_u^T \\ L_u & 0 \end{bmatrix} \begin{Bmatrix} U_2 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ X_\tau \end{Bmatrix} \end{aligned} \tag{25}$$

4.2 Discrete adjoint derivative

To evaluate the derivatives of E we use the adjoint method. The Lagrangian term is defined as follows:

$$\begin{aligned} &L\left(U_1, U_2, \lambda_1, \lambda_2, q_1, q_2, p_1, p_2, X_\eta, X_\tau\right) = E\left(U_1(X_\eta), U_2(X_\tau)\right) \\ &- \begin{Bmatrix} \lambda_1 \\ q_1 \end{Bmatrix}^T \left\{ \begin{bmatrix} K & L_m^T \\ L_m & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ p_1 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ U_m \end{Bmatrix} \right\} - \begin{Bmatrix} \lambda_2 \\ q_2 \end{Bmatrix}^T \left\{ \begin{bmatrix} K & L_u^T \\ L_u & 0 \end{bmatrix} \begin{Bmatrix} U_2 \\ p_2 \end{Bmatrix} - \begin{Bmatrix} F_2 \\ X_\tau \end{Bmatrix} \right\} \end{aligned} \tag{26}$$

where the vectors of Lagrange multipliers λ_i and q_i are virtual displacements and nodal forces, respectively. By virtue of (23) we obtain:

$$\frac{dL}{dX_\eta} = \frac{dE}{dX_\eta} \quad and \quad \frac{dL}{dX_\tau} = \frac{dE}{\partial X_\tau}$$

Differentiation of (26) with respect to the design parameters yields:

$$\frac{dL}{dX_\eta}=\frac{\partial E}{\partial U_1}\frac{dU_1}{dX_\eta}-\begin{Bmatrix}\lambda_1\\ q_1\end{Bmatrix}^T\left(\left\{\begin{bmatrix}K & L_m^T\\ L_m & 0\end{bmatrix}\begin{Bmatrix}\frac{dU_1}{dX_\eta}\\ \frac{dp_1}{dX_\eta}\end{Bmatrix}-\begin{Bmatrix}\frac{dF_1}{dX_\eta}\\ 0\end{Bmatrix}\right\}\right) \tag{27}$$

$$\frac{dL}{dX_\tau}=\frac{\partial E}{\partial U_2}\frac{dU_2}{dX_\tau}-\begin{Bmatrix}\lambda_2\\ q_2\end{Bmatrix}^T\left\{\begin{bmatrix}K & L_u^T\\ L_u & 0\end{bmatrix}\begin{Bmatrix}\frac{dU_2}{dX_\tau}\\ \frac{dp_2}{dX_\tau}\end{Bmatrix}-\begin{Bmatrix}0\\ I_{p\times p}\end{Bmatrix}\right\} \tag{28}$$

where $I_{p\times p}$ denotes the $p\times p$ square identity matrix. The derivative (27) can be written as follows:

$$\frac{\partial L}{\partial X_\eta}=\left\{\begin{Bmatrix}\frac{\partial E}{\partial X_\eta}\\ 0\end{Bmatrix}-\begin{Bmatrix}\lambda_1\\ q_1\end{Bmatrix}^T\begin{bmatrix}K & L_m^T\\ L_m & 0\end{bmatrix}\right\}\begin{Bmatrix}\frac{dU_1}{dX_\eta}\\ \frac{dp_1}{dX_\eta}\end{Bmatrix}+\begin{Bmatrix}\lambda_1\\ q_1\end{Bmatrix}^T\begin{Bmatrix}\frac{dF_1}{dX_\eta}\\ 0\end{Bmatrix} \tag{29}$$

where the implicit derivatives can be eliminated by selecting λ_1 so that the term in the round bracket is cancelled. This is achieved if λ_1 solves the following system:

$$\begin{bmatrix}K & L_m^T\\ L_m & 0\end{bmatrix}\begin{Bmatrix}\lambda_1\\ q_1\end{Bmatrix}^T=\begin{Bmatrix}\frac{\partial E}{\partial X_\eta}\\ 0\end{Bmatrix}=\begin{Bmatrix}K(U_1-U_2)\\ 0\end{Bmatrix} \tag{30}$$

System (30) is referred to as the adjoint problem for adjoint response λ_1 with the adjoint load $\partial E/\partial U_1$ and the adjoint Dirichlet boundary condition $\lambda_1=0$ on Γ_m. Once the λ_1 multipliers have been evaluated, the derivatives of E with respect to the specific parameters X_η are obtained by:

$$\frac{dE}{dX_\eta}=\begin{Bmatrix}\lambda_1\\ q_1\end{Bmatrix}^T\begin{Bmatrix}\frac{dF_1}{dX_\eta}\\ 0\end{Bmatrix}=\begin{Bmatrix}\lambda_1\\ q_1\end{Bmatrix}^T\begin{Bmatrix}I_{p\times p}\\ 0\end{Bmatrix}=\lambda_1^T|_{\Gamma_u} \tag{31}$$

Derivative (28) can be written as follows:

$$\frac{dL}{dX_\tau}=\left\{\begin{Bmatrix}\frac{\partial E}{\partial U_2}\\ 0\end{Bmatrix}-\begin{Bmatrix}\lambda_2\\ q_2\end{Bmatrix}^T\begin{bmatrix}K & L_u^T\\ L_u & 0\end{bmatrix}\right\}\begin{Bmatrix}\frac{dU_2}{dX_\tau}\\ \frac{dp_2}{dX_\tau}\end{Bmatrix}+\begin{Bmatrix}\lambda_2\\ q_2\end{Bmatrix}^T\begin{Bmatrix}0\\ I_{p\times p}\end{Bmatrix} \tag{32}$$

In this case λ_2 is defined so that the first term in round brackets in equation (32) is cancelled. This is achieved if λ_2 solves the following system:

$$\begin{bmatrix}K & L_u^T\\ L_u & 0\end{bmatrix}\begin{Bmatrix}\lambda_2\\ q_2\end{Bmatrix}=\begin{Bmatrix}\frac{\partial E}{\partial U_2}\\ 0\end{Bmatrix}=\begin{Bmatrix}K(U_2-U_1)\\ 0\end{Bmatrix} \tag{33}$$

System (33) is referred to as the adjoint problem for adjoint response λ_2 with adjoint load $\partial E / \partial U_2$ and adjoint Dirichlet boundary condition $\lambda_2=0$ on Γ_u. Once the λ_2 multipliers have been evaluated, q_2 can be expressed as follows:

$$L_u^T q_2 = \frac{\partial E}{\partial X_\tau} - K\lambda_2 = K(U_2 - U_1) - K\lambda_2 \tag{34}$$

Then, the derivatives of E with respect to the specific parameters X_τ are obtained by:

$$\frac{dE}{dX_\tau} = \begin{Bmatrix} \lambda_2 \\ q_2 \end{Bmatrix}^T \begin{Bmatrix} 0 \\ I_{p\times p} \end{Bmatrix} = L_u^T q_2 \mid_{\Gamma_u} = K\left(U_2 - U_1 - \lambda_2\right) \mid_{\Gamma_u} \tag{35}$$

Formulae (30), (33), (31) and (35) represent the discrete counterpart of the continuum results (17), (15), (18) and (16). The adjoint method requires the solution of two adjoint problems (30) and (33) for each response functional E. It is efficient when the number of functionals and constraints is low compared to the number of design parameters, which is clearly the case here as the design parameters are fields defined on a part of the boundary of the mesh.

4.3 Optimization procedure

By using the quadratic nature of E, the conjugate gradient algorithm could conceivably be used, although the constant Hessian matrix is never explicitly built. Each iteration of the algorithm involves the solution of four linear systems: two direct problems defined in (23) and two adjoint problems defined by (30) and (33). The quadratic energy functional can be written as follows:

$$\begin{aligned} &E(X) = \frac{1}{2} X^T H X + X^T B + C \\ &with \quad X^T = \left\{X_\eta^T, X_\tau^T\right\} \end{aligned} \tag{36}$$

where H is the Hessian which depends only on the geometric and material data, B and C depend on the geometric and material data and also on the measured boundary data (p_m, u_m). Due to the ill-posedness of the Cauchy problem, the Hessian matrix is only positive semi-defined, thus the convergence of the conjugate gradient is slow. In order to optimize the computational cost we adopt the Trust Region Method (TRM) which consists in solving the following subproblem for each iteration k:

$$\min\left\{Q(S_k) = \frac{1}{2} S_k^T H S_k + S_k^T \nabla E_{k-1} \;\; such\ that\ \left\|S_k^T\right\| \le \delta\right\} \tag{37}$$

where, $Q(S_k)$ is the quadratic approximation of $E(X_{k-1}+S_k)$, $\nabla E_{k-1} = HX_{k-1} + B$ is the energy functional gradient computed at the preceding iteration $(k-1)$, δ is a positive scalar, it is the size of the trust region and reflects the confidence in the second order model. As the energy functional E is quadratic $Q(S_k)$ is an exact approximation of $E(X_{k-1}+S_k)$, and the optimal search direction is Newtonian: $S_k = -H^{-1}\nabla E_{k-1}$. However, the Hessian matrix is singular and the gradient depends on the overspecified data; consequently, this search direction is not well defined. The search direction is defined by solving subproblem (37). The Lagrangian functional associated with this problem is:

$$L(X,\lambda) = \frac{1}{2} S_k^T H S_k + S_k^T \Delta E_{k-1} + \frac{\lambda}{2} S_k^T S_k \tag{38}$$

This quadratic function can be interpreted as a regularized quadratic model for E around X_{k-1}, and its exact solution is given by:

$$\left(H + \lambda I\right) S_k = -\nabla E_{k-1} \tag{39}$$

The Trust Region algorithm used in this study is that implemented in the *optimization toolbox* of Matlab. The approximation approach followed in the Optimization Toolbox is to restrict the Trust-Region Subproblem to a two-dimensional subspace (see Byrd *et al* and Coleman *et al*).

5 Numerical Results

To illustrate the method, we present an example that concerns the identification of contact pressure in an internal boundary of a two-dimensional elastic body. It should be noted here that the contact problem is nonlinear. However the data completion problem must be posed for that part of the domain where linear elasticity exists.

We consider the following direct problem: a plate with an internal hole in contact with a disk as shown in figure 1. The contact pressure has an elliptical distribution across the contact zone of width *2b*. We assume that the overspecified data are measured on the outer boundary Γ_m. In this study these data are generated numerically by solving the direct problem with the following data:

- Plate: $L = 2\ m$, $R_1 = 0.5\ m$ and $thickness = 0.01\ m$. The material is Aluminum with Young modulus $E = 70000\ MPa$ and Poisson coefficient $\nu = 0.31$.
- Disk: $R_2 = 0.495\ m$ and $thickness = 0.01\ m$. The material is Aluminum with Young modulus $E = 70000\ MPa$ and Poisson coefficient $\nu = 0.31$.
- $F = 0$ and $Q = 100000\ N$. The half contact width $b \approx 0.128\ m$.

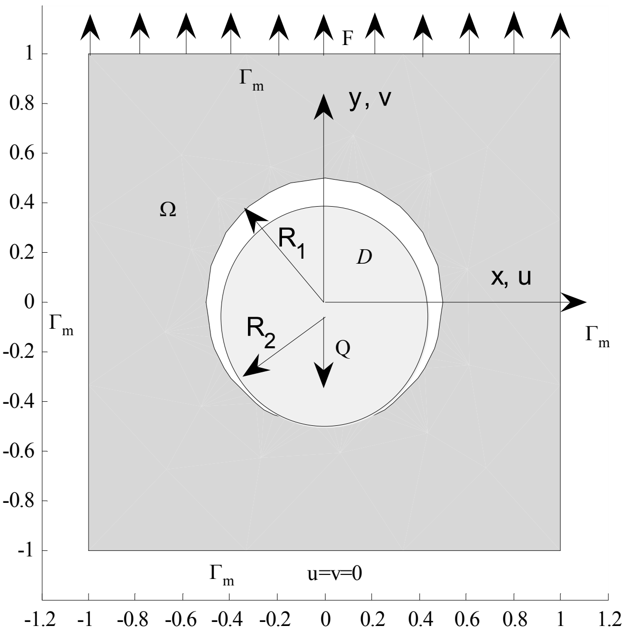

Fig. 2: Geometric data of the real problem.

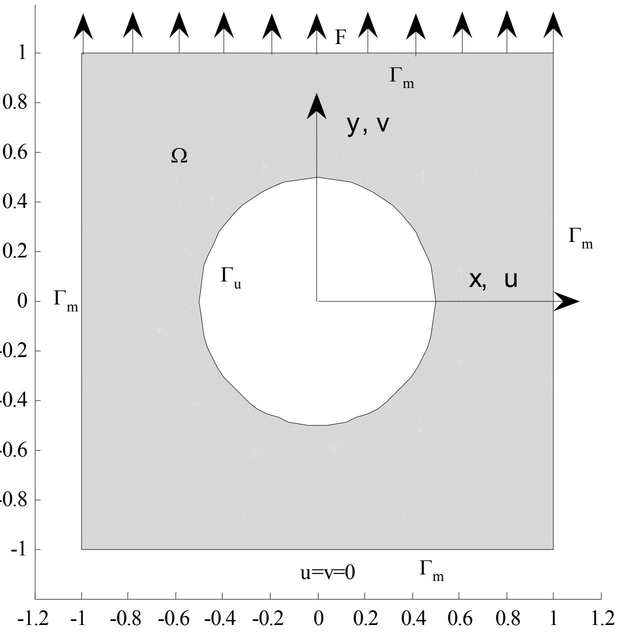

Fig. 3: Geometric data used in the Cauchy problem.

The contact width b and the pressure values are evaluated by using Hertz contact theory, see Johnson (1985). The data completion problem is solved using the geometric model as shown in figure 3. Thus, the overspecified data extracted from the direct problem are applied on this model by ignoring the disk in contact with the plate and force Q. The results are performed with finite elements of class C^o. The mesh of Ω is regular and consists of triangular elements with linear interpolation. It is characterized by 106 nodes on Γ_u. Figure 4 shows the reconstructed displacement fields of problems (3) and (4) compared with the *exact* one, indexed with e, obtained by solving the direct problem. We observe that they are in good agreement with the real one and also note that only fields u_1 and u_e are merged, field u_2 being translated with respect to u_1 as predicted by the properties of the E function in the first part of the paper. Figure 5 shows the reconstructed internal normal and tangential pressure compared with the real values. In spite of the very sharp variation of the surface tractions, the reconstruction remains in good agreement with the real values. Figure 6 shows the distribution map of energy error density over the domain, which appears uniform.

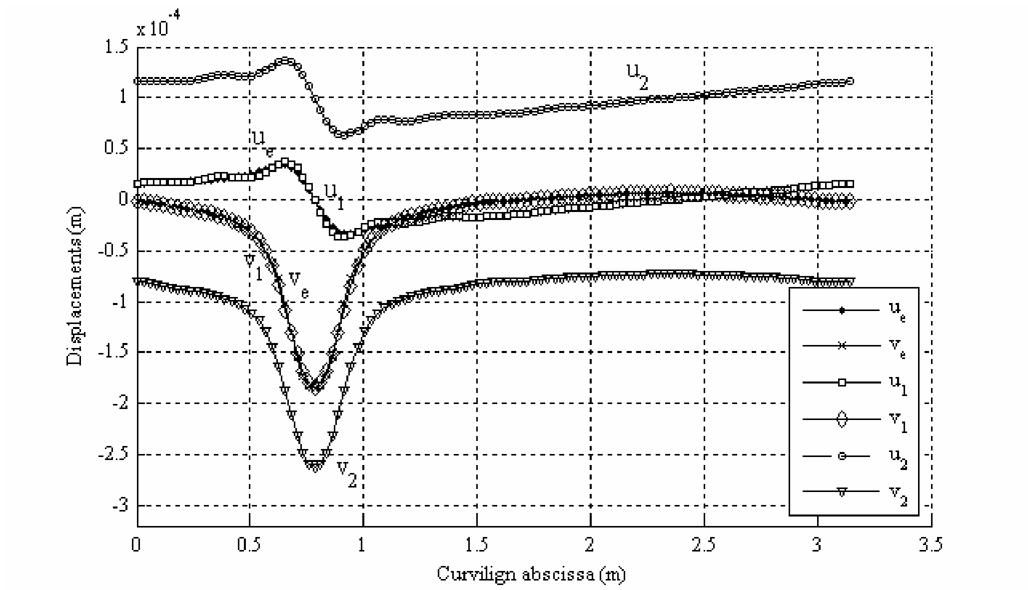

Fig. 4: Identified displacement fields (u_1, v_1) and (u_2, v_2) compared to the real fields (u_e, v_e) on Γ_u

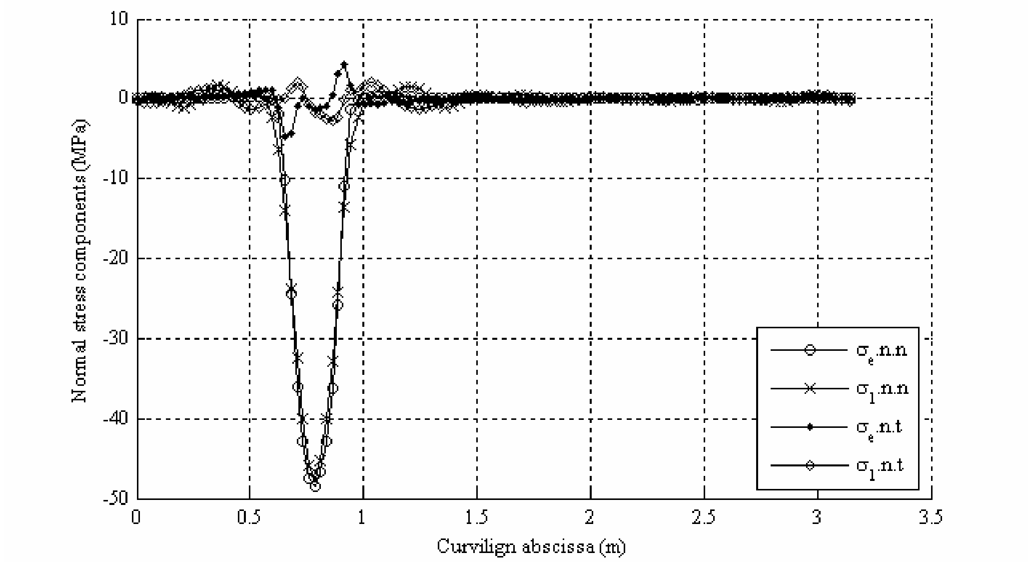

Fig. 5: Normal and tangential components of the internal pressure compared to the real pressure on Γ_u.

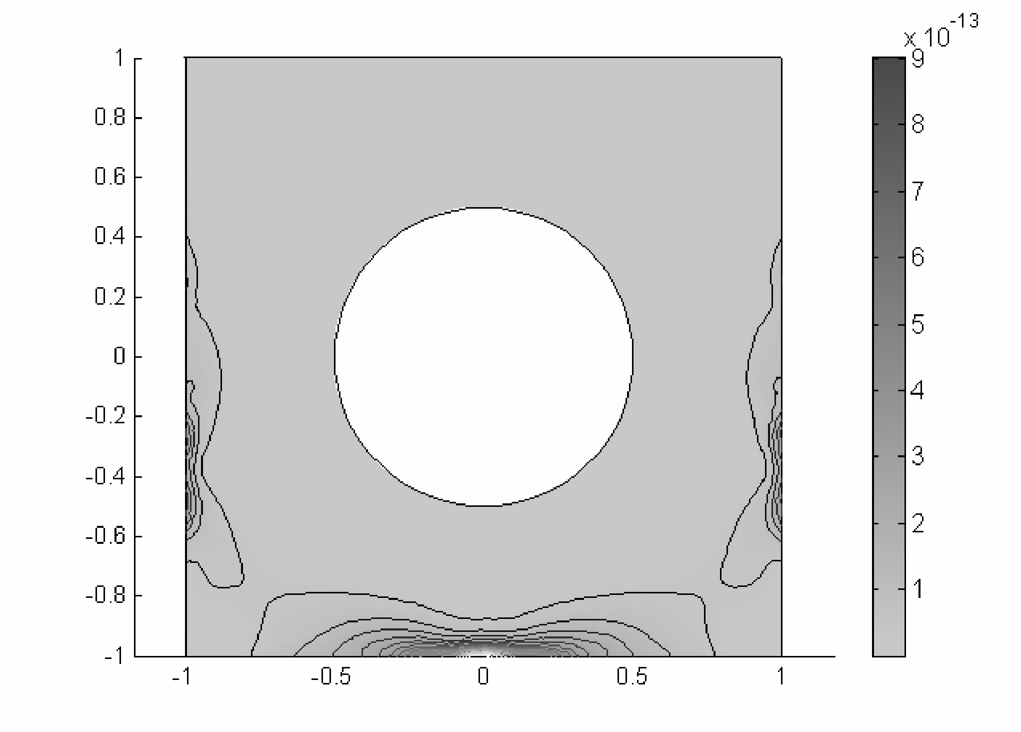

Fig. 6: Distribution map of energy error density.

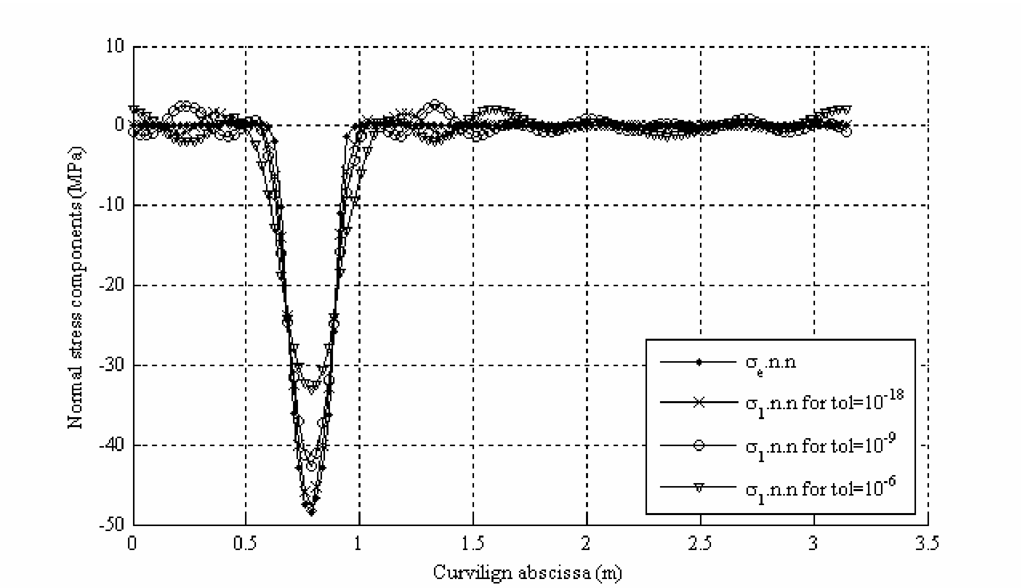

Fig. 7: Normal component of the internal pressure for different values of the optimization stopping criterion *Tol*.

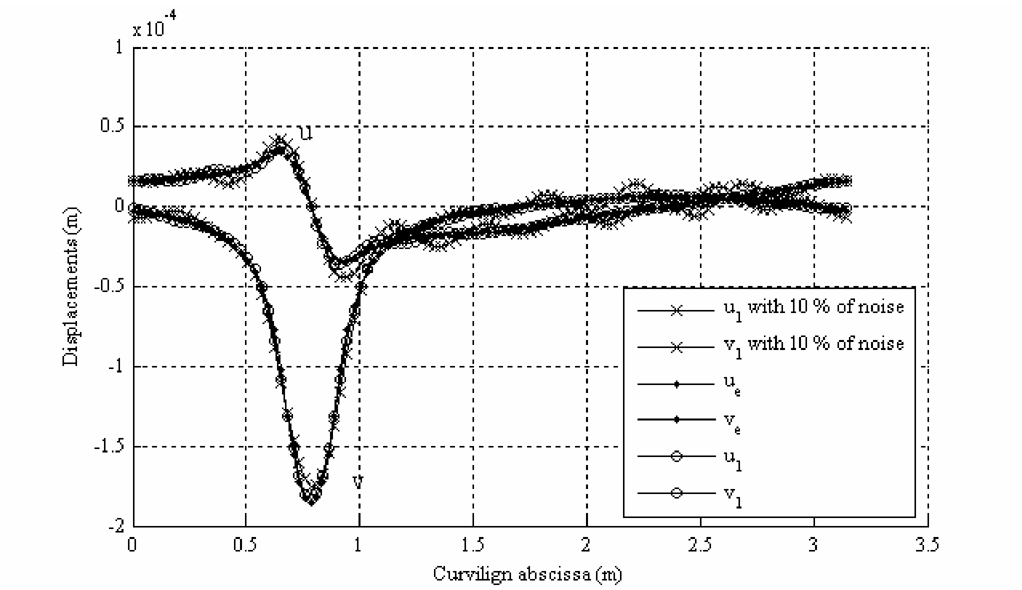

Fig 8: Reconstructed displacement components with 10% noise on the Dirichlet data.

The convergence of the optimization procedure was achieved with only 34 iterations for a stopping criterion equal to 10^{-18}. The results presented are obtained with this tolerance. Figure 7 shows the distribution of the normal component of the internal pressure identified as a function of the stopping criterion values *Tol*. Obviously, the choice of the criterion influences the quality of the field identified. However, it can be noted that with a criterion value equal to 10^{-6} (satisfied only with 3 iterations) the contact pressure is identified in a reasonable way with reasonable precision.

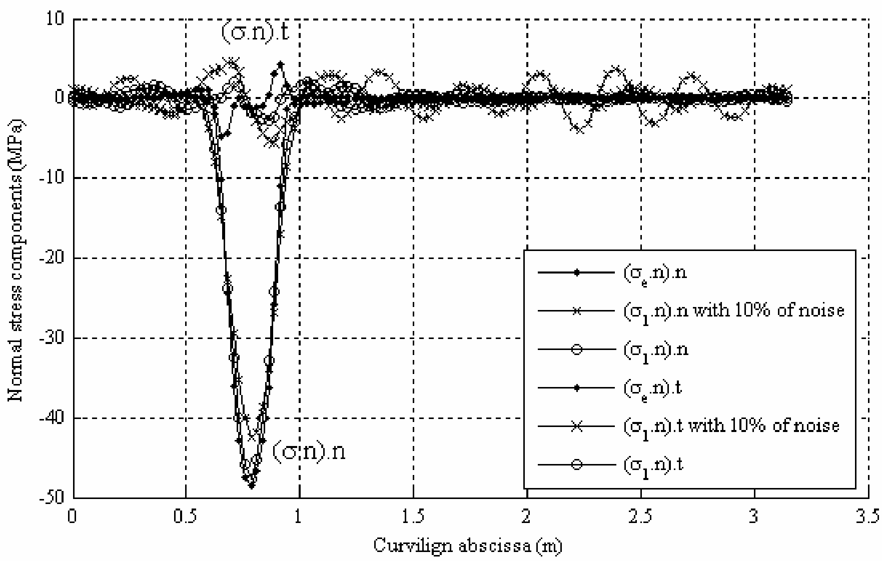

Fig. 9: Reconstructed pressure components with 10% of noise.

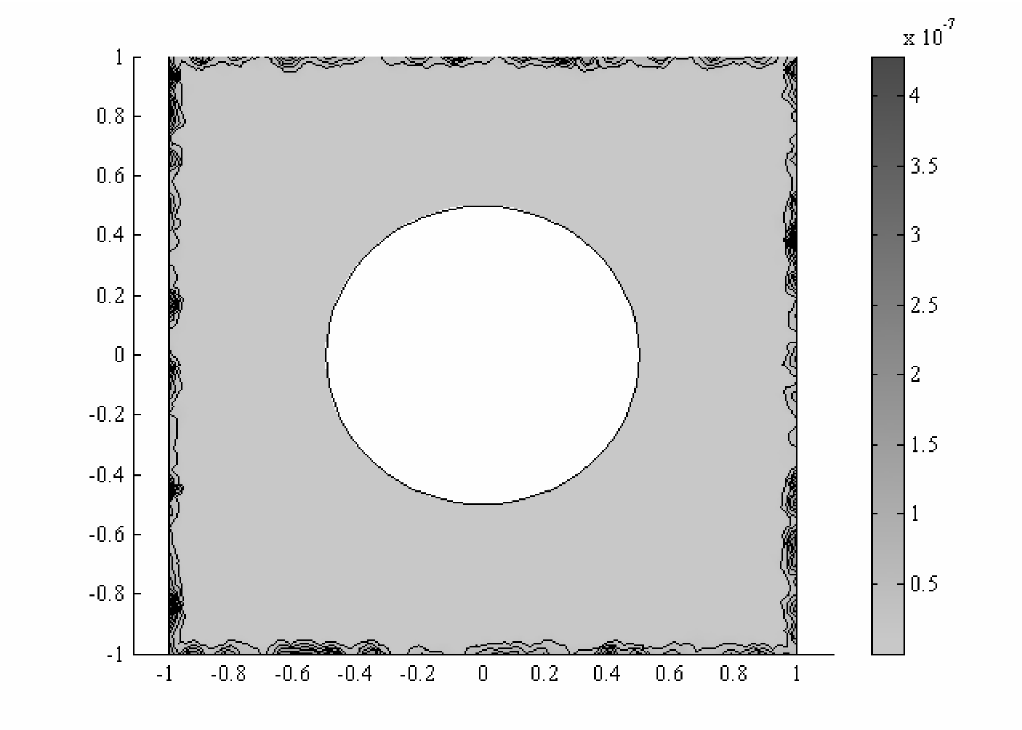

Fig. 10: Distribution map of the energy error density obtained with noisy data (10%)

The following table shows the performance of three optimization algorithms: the conjugate Gradient algorithm (CG), the alternating direction minimization method of Kozlov *et al* and the Trust Region Method used in this work. The first column indicates the stopping criterion value used for the Trust Region method. The second column indicates the energy error function value obtained. The following columns indicate the number of iterations needed to achieve convergence. The stopping criterion used for the CG and Kozlov algorithms is the value of the energy error functional indicated in the second column. It appears clearly that the Trust Region Method provides the best performance as shown in paragraph (4.3). It should be noted that in the two first cases optimization is terminated with the relative function value changing by less than *Tol*. However, in the last case optimization is terminated with the norm of the current step at less than 10^{-6}.

Tol	*E*	*GC*	*Kozlov et al*	*TRM*
10^{-6}	$1.612\ 10^{-9}$	347	76	3
10^{-9}	$9.713\ 10^{-12}$	1114	3976	6
10^{-18}	$5.510\ 10^{-14}$	3657	25000	34

Table 1: Comparison of performances for the algorithms used in this study.

To evaluate the robustness of the method with respect to imperfect data, the fields on the external part of the boundary of the solid were corrupted with a stochastic noise. Figures 8 and 9 show the recovered lacking data with a noise amplitude of 10% of the maximum values of the noise free fields. These results were obtained with only five iterations and a stopping criterion of $Tol=10^{-9}$. It should be noted that the reconstruction remains robust for reasonable levels of noise despite the ill-posedness of the problem addressed. Figure 10 shows the distribution map of the energy error density over the domain, which appears to be uniform.

6. Conclusion

The problem of expanding measured surface fields inside a solid body of known physical properties has been addressed by implementing a new energy based variational approach. Compared with other approaches it seems that the error energy approach, associated with an effective minimization method leads to an efficient tool as few iterations are needed to achieve convergence. Reasonable robustness with respect to noise, within the level encountered in real measurements, has been reported. Its extension to nonlinear constitutive equations is possible within the convex setting but the energy error will no longer be the energy of the difference of the two displacement fields. Three dimensional applications and extension to more realistic situations (such as those in which only tangential displacement is measured on a traction free boundary) are currently being studied.

The functional used in this work falls within the general framework of constitutive law errors. Consequently it can be used for identifying elastic parameters. The problem posed is not in its usual form due to the lack of data at the boundaries and contains more unknown variables, i.e. the lacking boundary conditions and parameters of the material. However, it is possible to process it by adding a loop in the optimization process to deal with the variable hierarchically. This problem is now the subject of studies in progress.

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